Functions

Definition A *function* is a correspondence from a set, called its *domain*, to another set, called its *codomain*, that associates to each element of the domain exactly one element of the codomain.

It is common to represent an element of the domain with a variable, say x, and an element of the codomain another variable, say y. Since each value of x determines a corresponding value of y, we say that "y depends on x". We also call x the *independent variable* and y the *dependent variable.*

Notation and Terminology

We use the following notation

$$f : X \longrightarrow Y$$

to denote the fact that f is a function with domain X and codomain Y.

For every $x \in X$, we denote the element in Y corresponding to x by $f(x)$. $f(x)$ is called the image of x under f. The *range* of f, denoted by $f(X)$, is defined to be the set elements of the codomain Y that are images under f. Note that $f(X) \subseteq Y$ always holds, but the functions for which $f(X) \subsetneq Y$.

Examples

1. $P := \{\text{persons}\}$, $M := \{\text{mothers}\}$.
   Let $f : P \longrightarrow M$ and $f(x) = x$'s mother. Let $g : M \longrightarrow P$ and $g(y) = y$'s offspring.
   - Then f is a function from P to M, whereas g is not a function from M to P.
   - Every element in M corresponds to some element in P, hence $f(P) = M$.
2. $P := \{\text{persons}\}$, $W := \{\text{women}\}$.
   Let $h : P \longrightarrow W$ and $h(x) = x$'s mother.
   - Then h is a function from P to W,
   - However, now, there are elements in W that are not images under f. Hence $h(P) \subsetneq W$.

Mathematical Examples

1. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $f(x) = x^2$. That is, both the domain and range of f are $\mathbb{R}$ and the "action" of f is to map any given real number x to the real number x^2. Note that the range of f is $f(\mathbb{R}) = [0, \infty) \subsetneq \mathbb{R}$.
2. Let $g : [0, \infty) \longrightarrow \mathbb{R}$ and $g(y) = -y$. Then the domain of g is $[0, \infty)$, the set of all non-negative real numbers. The codomain of g is $\mathbb{R}$, while the range of g is the set of all non-positive real numbers, i.e. $g([0, \infty)) = (-\infty, 0] \subsetneq \mathbb{R}$.

Exercises For questions 1, 2 and 3, determine the domain, codomain and range of the given function.

1. $f : [0, \infty) \longrightarrow \mathbb{R}$ and $f(x) = -x$.
2. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $f(x, y) = -(xy)^2$. Here, $\mathbb{R}^2$ denotes the set of all ordered pairs (x, y) of real numbers.
3. $f : [0, 1] \cup [2, 3] \longrightarrow \mathbb{R}$ and
$$f(x) = \begin{cases} 2x, & x \in [0, 1] \\ -3x, & x \in [2, 3] \end{cases}$$
4. Suppose the action of f is given by $f(x) = \sqrt{2x + 1}$, where x is a real number. What is the largest possible domain of f?
   Hint: For what values of $x \in \mathbb{R}$ is $f(x)$ undefined? The largest possible domain of f is simply the complement of that set in $\mathbb{R}$.
5. Suppose the action of f is given by $f(x) = \frac{2x+3}{3x+4}$, where x is a real number. What is the largest possible domain of f?

6. Let $S : \mathbb{N} \longrightarrow \mathbb{Z}$ and $S(n) = \frac{1}{2}n(n-1)$.

   a) What are the domain and codomain of S?

   b) Evaluate

      i) $S(1)$
      ii) $S(2)$
      iii) $S(n-1)$
      iv) $S(n) - S(n-1)$

7. Is y a function of $x \in [0, \infty)$ if y is the solution to the equation $y^2 = x$?
   Hint: Consider the case when $x = 4$. What is y? Or, how many possible values for y are there? Is **(I)** violated?