# Operations on Functions



We wish to obtain new functions from given ones. For example, we may attempt to define the sum, difference, product and quotient of two given real-valued functions.

Suppose we are given two real-valued functions $f : X \longrightarrow \mathbb{R}$ and $g : Y \longrightarrow \mathbb{R}$. It is natural to define a the sum $f + g$ of $f$ and $g$ via

$$(f + g)(x) = f(x) + g(x).$$

But immediately, for the above to be well defined, the domain of $f$ must be the same as the domain of $g$.

## Definition

Let $f, g : X \longrightarrow \mathbb{R}$ be two given functions from the set $X$ to the real numbers. We define the

- the *sum* $f + g : X \longrightarrow \mathbb{R}$ of $f$ and $g$ by
$$(f + g)(x) = f(x) + g(x).$$
- the *difference* $f - g : X \longrightarrow \mathbb{R}$ of $f$ and $g$ by
$$(f - g)(x) = f(x) - g(x).$$
- the *product* $fg : X \longrightarrow \mathbb{R}$ of $f$ and $g$ by
$$(fg)(x) = f(x)g(x).$$
- the *quotient* $\frac{f}{g} : X \longrightarrow \mathbb{R}$ of $f$ and $g$ by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$, for every $x \in X$. If $g(x) = 0$ for any $x \in X$, then $\frac{f}{g}$ is not defined.

## Examples

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be $f(x) = x^2$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be $g(x) = x$. Then, $(f + g)(x) = x^2 + x$, $(f - g)(x) = x^2 - x$, $(fg)(x) = x^3$, and their domains are all $\mathbb{R}$. However, $\frac{f}{g}$ is not defined since $g(0) = 0$.
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be $f(x) = x$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be $g(x) = x^2+1$. Then, $(f+g)(x) = x+x^2+1$, $(f-g)(x) = x-x^2-1$, $(fg)(x) = x^3+x$. Moreover, $\frac{f}{g}$ is well-defined and $(\frac{f}{g})(x) = \frac{x}{x^2+1}$, since $g$ is never 0. The domains of these functions are all $\mathbb{R}$.
3. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be $f(x) = \sqrt{x}$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be $g(x) = x$. Then *none* of $f + g$, $f - g$, $fg$, $\frac{f}{g}$ can be defined since for $f(x)$ is not defined for negative values of $x$.

## Definition (Composition of two functions)

Let $g : X \longrightarrow Y$ and $f : W \longrightarrow Z$ be two given functions.
Suppose $g(X) \subseteq W$. Then, we may define the *composition*, $f \circ g : X \longrightarrow Z$ by

$$(f \circ g)(x) = f(g(x)).$$

## Examples

1. Let $P :=$ {persons}, $M :=$ {mothers}, and $F :=$ {fathers}.
Let $g : P \longrightarrow M$ be $g(x) = x$'s mother, and $f : P \longrightarrow F$ be $f(y) = y$'s father.
    - Then $f \circ g$ is well-defined since $g(P) = M \subseteq P = \text{domain}(f)$.
    - And, $f \circ g : P \longrightarrow F$ is given by
$$(f \circ g)(x) = x\text{'s maternal grandfather.}$$

- Note also that we have $f(P) = F$ but $(f \circ g)(P) \subsetneq F$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be $f(x) = x^4$, and $g : [0, \infty) \longrightarrow \mathbb{R}$ be $g(y) = \sqrt{y}$.
   - Then, $f \circ g$ is well-defined since $g(\mathbb{R}) = [0, \infty) \subseteq \mathbb{R} = \text{domain}(f)$.
   - And, $f \circ g : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^4 = x^2.$$

3. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be $f(x) = \sqrt{x}$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be $g(y) = -y$.
   - Then, $f \circ g$ is not defined since $g(\mathbb{R}) = \mathbb{R} \nsubseteq [0, \infty) = \text{domain}(f)$.
   - In particular, $g(1) = -1$ and hence $f(g(1)) = \sqrt{-1}$ is not in $\mathbb{R}$.

## Exercises

1. Do Problems 1.2.23 to 1.2.30 in the textbook.
2. Suppose $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$. Prove that
   a) both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are well-defined.
   b) for every $x \in A$, we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$.
   [This means that the two functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are in fact the same.]