We wish to obtain new functions from given ones. For example, we may attempt to define the sum, difference, product and quotient of two given real-valued functions.

Suppose we are given two real-valued functions \( f : X \rightarrow \mathbb{R} \) and \( g : Y \rightarrow \mathbb{R} \). It is natural to define a the sum \( f + g \) of \( f \) and \( g \) via

\[
(f + g)(x) = f(x) + g(x).
\]

But immediately, for the above to be well defined, the domain of \( f \) must be the same as the domain of \( g \).

**Definition**

Let \( f, g : X \rightarrow \mathbb{R} \) be two given functions from the set \( X \) to the real numbers. We define the

- the sum \( f + g : X \rightarrow \mathbb{R} \) of \( f \) and \( g \) by
  \[
  (f + g)(x) = f(x) + g(x).
  \]
- the difference \( f - g : X \rightarrow \mathbb{R} \) of \( f \) and \( g \) by
  \[
  (f - g)(x) = f(x) - g(x).
  \]
- the product \( fg : X \rightarrow \mathbb{R} \) of \( f \) and \( g \) by
  \[
  (fg)(x) = f(x)g(x).
  \]
- the quotient \( \frac{f}{g} : X \rightarrow \mathbb{R} \) of \( f \) and \( g \) by
  \[
  \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)},
  \]
  provided \( g(x) \neq 0 \), for every \( x \in X \). If \( g(x) = 0 \) for any \( x \in X \), then \( \frac{f}{g} \) is not defined.

**Examples**

1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( f(x) = x^2 \), and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( g(x) = x \). Then, \( (f + g)(x) = x^2 + x \), \( (f - g)(x) = x^2 - x \), \( (fg)(x) = x^3 \), and their domains are all \( \mathbb{R} \). However, \( \frac{f}{g} \) is not defined since \( g(0) = 0 \).

2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( f(x) = x \), and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( g(x) = x^2 + 1 \). Then, \( (f + g)(x) = x + x^2 + 1 \), \( (f - g)(x) = x - x^2 - 1 \), \( (fg)(x) = x^3 + x \). Moreover, \( \frac{f}{g} \) is well-defined and \( \left( \frac{f}{g} \right)(x) = \frac{x}{x^2 + 1} \), since \( g \) is never 0. The domains of these functions are all \( \mathbb{R} \).

3. Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be \( f(x) = \sqrt{x} \), and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( g(x) = x \). Then none of \( f + g \), \( f - g \), \( fg \), \( \frac{f}{g} \) can be defined since for \( f(x) \) is not defined for negative values of \( x \).

**Definition (Composition of two functions)**

Let \( g : X \rightarrow Y \) and \( f : W \rightarrow Z \) be two given functions.

Suppose \( g(X) \subseteq W \). Then, we may define the composition, \( f \circ g : X \rightarrow Z \) by

\[
(f \circ g)(x) = f(g(x)).
\]

**Examples**

1. Let \( P := \{ \text{persons} \} \), \( M := \{ \text{mothers} \} \), and \( F := \{ \text{fathers} \} \).

   Let \( g : P \rightarrow M \) be \( g(x) = x \)'s mother, and \( f : P \rightarrow F \) be \( f(y) = y \)'s father.
   - Then \( f \circ g \) is well-defined since \( g(P) = M \subseteq P = \text{domain}(f) \).
   - And, \( f \circ g : P \rightarrow F \) is given by
     \[
     (f \circ g)(x) = x \text{'s maternal grandfather}.
     \]
• Note also that we have \( f(P) = F \) but \( (f \circ g)(P) \not\subseteq F \).

2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( f(x) = x^4 \), and \( g : [0, \infty) \rightarrow \mathbb{R} \) be \( g(y) = \sqrt{y} \).

• Then, \( f \circ g \) is well-defined since \( g(\mathbb{R}) = [0, \infty) \subseteq \mathbb{R} = \text{domain}(f) \).
• And, \( f \circ g : [0, \infty) \rightarrow \mathbb{R} \) is given by
  \[
  (f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^4 = x^2.
  \]

3. Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be \( f(x) = \sqrt{x} \), and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( g(y) = -y \).

• Then, \( f \circ g \) is not defined since \( g(\mathbb{R}) = \mathbb{R} \not\subseteq [0, \infty) = \text{domain}(f) \).
• In particular, \( g(1) = -1 \) and hence \( f(g(1)) = \sqrt{-1} \) is not in \( \mathbb{R} \).

Exercises

1. Do Problems 1.2.23 to 1.2.30 in the textbook.

2. Suppose \( f : A \rightarrow B, g : B \rightarrow C, \) and \( h : C \rightarrow D \). Prove that
   a) both \( h \circ (g \circ f) \) and \( (h \circ g) \circ f \) are well-defined.
   b) for every \( x \in A \), we have \( (h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x) \).
   [This means that the two functions \( h \circ (g \circ f) \) and \( (h \circ g) \circ f \) are in fact the same.]