Solutions with Unbounded Derivatives

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Solving a variational problem, we look for the solution $u(x)$ in the class of differentiable functions. But the set of differentiable functions is open. The limit of a minimizing sequence of differentiable functions may be discontinuous.

Example 0.1 The sequence of continuous functions $f_n(x) = \frac{1}{1+|x|^n}$ has a discountinuous limit

$$\lim_{n\to\infty} \frac{1}{1+x^n} = \begin{cases} 1 & |x| < 1 \\ 1/2 & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

The optimization problems tend to reveal an extremal behavior of minimizers and their solutions may show unexpected features. Here we introduce and discuss variational problems with discontinuous solutions that have unbounded derivatives.

1 Example of discontinuous extremal (Problem H)

1.1 Euler equation

Consider the minimization problem

$$I_0 = \min_{u(x)} I(u), \quad I(u) = \int_{-1}^{1} x^2 u'^2 dx, \quad u(-1) = -1, \quad u(1) = 1. \tag{1}$$

We observe that $I(u) \geq 0 \ \forall u$, and therefore $I_0 \geq 0$. The Lagrangian $L = x^2 u'^2$ is a convex function of u', so the Weierstrass condition is satisfied.

The Euler equation $\frac{d}{dx}(x^2 u') = 0$ admits the first integral

$$\frac{\partial L}{\partial u'} = 2x^2 u' = C \quad \text{and } u' = \frac{C}{2x^2}$$

If $C \neq 0$, the value of $I(u)$ is infinity, because the Lagrangian becomes

$$x^2 u'^2 = \frac{C^2}{4x^2}$$

and the integral of the Lagrangian diverges if $C \neq 0$.

We conclude that $C = 0$ which implies that $u'(x) = 0$ if $x \neq 0$. Accounting for the boundary conditions, we find

$$u(x) = \begin{cases} -1 \text{ if } x < 0 \\ \ \ 1 \text{ if } x > 0 \end{cases}$$

and $u(0)$ is not defined.

We arrive at an unexpected result that contradicts the assumptions of differentiability of $u(x)$ used in the derivation of the Euler equation: the extremal $u(x)$ is discontinuous and u' exists only in the generalized sense:

$$u_0(x) = -1 + 2H(x), \quad x \neq 0 \tag{2}$$

where $H(x)$ is the Heavyside function,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Weierstrass-Erdmann condition The Weierstrass-Erdmann condition

$$\left[\frac{\partial L}{\partial u'}\right]_-^+ = \left[x^2 u'\right]_-^+ = 0$$

shows that Lagrangian L is undetermined when $x = 0$ but the equality is satisfied if it is understood in the sense of distributions.

1.2 Minimizing sequence

Without a reference to the Euler equation, we build a minimizing sequence of differentiable functions that tends to (2). Let $u^\epsilon(x)$, $\epsilon > 0$ be

$$u^\epsilon(x) = \begin{cases} -1 & \text{if} \quad x \in \left[-1, -\frac{1}{\epsilon}\right] \\ 1 & \text{if} \quad x \in \left[\frac{1}{\epsilon}, 1\right] \\ \frac{x}{\epsilon} & \text{if} \quad x \in \left[-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right] \end{cases}$$

The derivative $[u^\epsilon]'$ is

$$[u^\epsilon]'(x) = \begin{cases} 0 & \text{if} \quad |x| \geq \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & \text{if} \quad |x| < \frac{1}{\epsilon} \end{cases}$$

The cost of the problem

$$I(u^\epsilon) = \int_{-1}^{1} x^2 (u^\epsilon)'^2 dx = \frac{1}{\epsilon^2} \int_{-\epsilon}^{\epsilon} x^2\, dx = \frac{2}{3}\epsilon$$

goes to zero, $I(u^\epsilon) \to 0$ when $\epsilon \to 0$. The sequence $\{u^\epsilon\}, \epsilon \to 0$ is a minimizing sequence for $I(u)$ which is nonnegative, $I(u) \geq 0\ \forall u$, for any trial function u.

This example again illustrates that *any variational problem has a solution, if the word "solution" is properly understood. /David Hilbert/* The differentiable solution does not exist but one can build a minimizing sequence and take its limit as the solution.

2 Minimal surface of revolution

2.1 Euler equaton

Another classic example of discontinuous minimizer is the classical problem of the minimal surface (Lagrange 1762). It is required to find a surface of minimal area, that is supported by a given spacial curve or curves. Here, we formulate it for the surface of revolution: Minimize the area of the surface of revolution supported by two parallel circumferences of two circles with the centers at the axis $r = 0$.

If is convenient to use cylindrical coordinates $z, r(z)$. According to the calculus, the area J of the surface of revolution is

$$J = 2\pi \int_{z_2}^{z_1} r\sqrt{1 + r'^2}\, dz \tag{3}$$

where $r' = \frac{dr}{dz}$, z_1 and z_2 are coordinated of the centers of the supporting circles. This problem is formally equivalent to a special case of the geometric optics problem, corresponding to the slowness $\psi(r) = r$ or to the speed $v = \frac{1}{r}$.

The Lagrangian

$$F = r\sqrt{1 + r'^2}$$

admits the first integral

$$r' \frac{\partial F}{\partial r'} - F = \frac{r}{\sqrt{1 + r'^2}} = c \tag{4}$$

Solving for $r' = \frac{dr}{dz}$ and separating variables, we find

$$dz = c \frac{dr}{\sqrt{r^2 - c^2}}$$

and

$$z = c \int \frac{dr}{\sqrt{r^2 - c^2}} = c \cosh^{-1}\left(\frac{r}{c}\right) + c_1$$

or, finally,

$$r(z) = c \cosh\left(\frac{z - c_1}{c}\right)$$

Constants c, c_1 are defined by the boundary conditions: radii of two supporting circumferences:

$$c \cosh\left(\frac{z_1 - c_1}{c}\right) = R_1, \quad c \cosh\left(\frac{z_2 - c_1}{c}\right) = R_2$$

Assume for clarity that the surface is supported by two equal circles of radius $R = R_1 = R_2$ that are located symmetric to OZ axis, $z_1 = -z_2$. Due to this symmetry, the surface is an even function of z, which implies $c_1 = 0$:

$$r(z) = c \cosh\left(\frac{z}{c}\right) \tag{5}$$

The minimal surface of revolution is a catenoid. The radius of the surface reaches minimum at the middle of the interval $[-z_1, z_1]$: $z = 0$, $r(0) = c$, as it is intuitively expected. The constant c can be found from the boundary condition that is a transendental equation for c:

$$r(z_1) = c \cosh\left(\frac{z_1}{c}\right) = R \tag{6}$$

Figure 1: Family of catenoids (5)

We compute the area of the minimal surface substituting $r(z)$ from (5) into the formula

$$J = 2(2\pi) \int_0^{z_1} (r\sqrt{1 + r'^2})\, dz.$$

We obtain, after calculation,

$$J = \pi c^2 \left[\sinh\left(\frac{2z_1}{c}\right) + \frac{2z_1}{c} \right]. \tag{7}$$

Figure 1 shows that two extremals pass through each point inside the cone, which means that there are two different stationary solutions. Indeed, equation (6) for c has two solutions. A direct calculation of the cost of the problem (7) is needed to determine which of these solution corresponds to the minimum of J. The examination shows that the less steep catenoid is the one with a smaller value of the objective functional. The detailed analysis of this phenomenon can be found in

http://mathworld.wolfram.com/MinimalSurfaceofRevolution.html

Family of extremals The family (5) of extremals $\frac{z}{c} = \cosh\left(\frac{z}{c}\right)$ forms a c-dependent family of functions varying only by simultaneously rescaling of the variables r and z. The ratio of the coordinates $(\frac{r}{c}, \frac{z}{c})$ on any curve of the family is independent on the scale c. Geometrically, this means that all curves of the family lie inside a triangular region which boundary is defined by a minimal value of that angle, or by the equation of the envelope of the family of function.

Recall that an envelope of the family of curves $f_c(x, y) = 0$ that depends on a parameter c satisfies the equations

$$\frac{\partial f_c(x, y)}{\partial c} = 0, \quad f_c(x, y) = 0$$

. which allow for excluding of the paprameter c

For the analysis of our problem it is convenient to change the parameter of the family c to $\frac{1}{c'}$ the family of extremals is In the considering case

$$f_{c'}(r, z) = 0 \quad \to \ c'r - \cosh(c'z) = 0 \tag{8}$$

We compute

$$\frac{df_{c'}(r, z)}{dc'} = 0 \quad \to \ r = z \sinh(c'z) \tag{9}$$

Divide (8) by (9) and call $c'z = p$. We find equation for p

$$p = \coth(p) \quad p = 1.1996789403....$$

From (9), we compute the slope of the cone:

$$\frac{r}{z} = \sinh p = 0.6627434193....$$

All extremal catenoids (5) lie within a triangle $\frac{r}{|z|} \geq 0.6627434193....$

Further analysis If the supporting circumferences are far away from each other, $\frac{R}{|z_1|} < 0.6627434193....$, none of the extremals passes through the boundary point. The problem does not have a solution. However, any minimal area is positive and the infimum is also a non-negative number that should correspond to a curve. What is wrong?

2.2 Goldschmidt solution

We implicitly assumed that the minimal surface of revolution is a differentiable function with finite tangent r' to the axis OZ of revolution. Therefore, we excluded curves for which $r'(z) = \infty$. These curves represent disks that can be a part of the optimal surface.

Another implicit assumption is that the radius is strictly positive. The set of positive numbers is open, the radius of the surface of revolution may tend to zero in an interval, and the surface area of the corresponding cylinder can be made arbitrarily small.

Accounting for these two factors results in another solution called Goldschmidt solution (it was found by Carl Wolfgang Benjamin Goldschmidt, in 1831).

Formally, the constant c in (4) can be zero; which implies that

$$\text{either } r = 0 \quad \text{or } r' = \infty$$

The minimal surface consists of two disks ($r' = \infty$) and an infinitesimal circular tube between them ($r = 0$). The surface area of the tube is neglectibly small and the cost J_G is equal to the double area of the circle: $J_G = 2\pi R^2$. Remarkable, that the cost is independent of the distance between circles.

Minimizing sequence As in the previous example,we can build a minimizing sequence of continuous functions $r_n(z)$, such that the limit corresponds to discontinuous limiting function. A function $r_\epsilon(z)$ in the minimizing sequence describes a cylinder of the radius ϵ joined with the cone with the angle arctan $\frac{2}{\epsilon}$ by the vertex. These two elements meet at the point $z = \kappa$.

$$r_\epsilon(z) = \begin{cases} \epsilon & \text{if } z \in (0, \kappa] \\ R - \frac{z_1 - z}{\epsilon} & \text{if } z \in (\kappa, z_1] \end{cases} \tag{10}$$

where $\kappa = z_1 - \epsilon R + \epsilon^2$.

The area A_ϵ is the sum of the areas $A_1 = 2\pi\epsilon\kappa$ of the cylinder and $A_2 = \pi R\sqrt{R^2 + H^2}$ of the cone. Since the height $H = z_1 - \kappa = \epsilon(R + \epsilon)$, we observe

$$A_1 \to 0, \quad A_2 = \pi R\sqrt{R^2 + \epsilon^2(R+\epsilon)^2} = \pi R^2 + O(\epsilon^2) \to \pi R^2, \quad \text{if } \epsilon \to 0$$

The Goldschmidt solution corresponds the discontinuous limit of this minimizing sequence of continuous functions $\{r_\epsilon\}$.

Stitched solutions The Goldschmidt solution is defined for any values of R, Z. A separate calculation is needed to find a region where the catenoid solution delivers a smaller value of the cost than the Goldschmidt solution. Direct comparison confirms that Goldschmidt solution corresponds to the smaller area than catenoid even is some part of the domain where both solutions are possible, see
http://mathworld.wolfram.com/MinimalSurfaceofRevolution.html

Fastest path: Harry Potter Chimney Transportation Strategy Geometric optics analogy suggests a physical interpretation of the Goldschmidt solution. The problem of a minimal surface is formally identical to the problem of the fastest path between two equally distant from OZ-axis points, if the speed $v = 1/r$ is inversely proportional to the distance to the axis OZ. The optimal path between the two close-by points lies along the arch of an appropriate catenoid $r = c\cosh(\frac{z}{c})$ that passes through the given endpoints; the path sags toward the OZ-axis where the speed is higher.

The optimal path between two far-away points is different: The travelers go straight to the OZ-axis where the speed is infinite, then they are transported instantly (infinitely fast) to the closest to the destination point at the axis, and then they go straight to the destination. This "*Harry Potter Chimney Transportation Strategy*" is optimal when the two supporting circles are far away from each other.

3 Discontinuous extremals

Now we analyze the previous problems from the analytical viewpoint without references to geometrical and physical interpretations. We demonstrated that in some variational problems the extremal $u(x)$ breaks the basic assumptions of continuity and admits discontinuities. Such solutions cannot be excluded because the set of differentiable functions is open, and a minimizing sequence of continuous functions may tend to a discontinuous limit.

The question is: What Lagrangians $F(x, u, u')$ can correspond to such solutions?

3.1 Lagrangians of linear growth with respect of u'

The Lagrangian $L_{ms} = \psi(x, u)\sqrt{1 + u'^2}$ in the geometric optics problems grows linearly with u' as $|u'| \to \infty$:

$$\lim_{|u'|\to\infty} \frac{L(x, u, u')}{|u'|} = \psi(x, u) \tag{11}$$

We show that the discontinuity of $u(x)$ for Lagrangians of linear growth adds a finite cost to the cost of the variational problem; therefore the jumps in the solutions are admissible.

Assume that a Lagrangian $L(x,u,u')$ has the property (11) where ψ is a bounded function, and consider a trajectory with a steep increase of u in the interval $[x_0, x_0+\epsilon]$:

$$\begin{aligned} u(x_0) &= u_1, \quad u(x_0+\epsilon) = u_2 \\ u'(x) &\approx \frac{u_2 - u_1}{\epsilon} \gg 1 \end{aligned}$$

The integral over the interval $[x_0, x_0+\epsilon]$ is

$$I_\epsilon = \int_{x_0}^{x_0+\epsilon} L(x,u,u')dx = \int_{x_0}^{x_0+\epsilon} \left(\frac{L(x,u,u')}{u'}\right) u'dx$$

Using (11) and smallness of the interval $[x_0, x_0+\epsilon]$ and continuity of $L(x,.,.)$ as a fi=unction of x, we approximate the Lagrangian :

$$\psi(x,u) = \psi(x_0,u) + O(\epsilon) \quad \forall x \in [x_0, x_0+\epsilon]$$

and the value of the integral I_ϵ becomes

$$I_\epsilon = I_{jump} + O(\epsilon^2), \quad I_{jump} = \int_{x_0}^{x_0+\epsilon} \psi(x_0,u)u'dx,$$

Rewriting $u'dx = du$ and correspondingly changing the limits of integration, we bring the last integral to the form

$$I_\epsilon = I_{jump} = \int_{u_1}^{u_2} \psi(x_0,u)\, du$$

When $\epsilon \to 0$, the contribution I_{jump} of the discontinuity remains finite. Hence, in some problems with Lagrangians of linear growth optimal solution may have discontinuities.

Example 3.1 In the Goldschmidt solution, $\psi = r$ and

$$I_{jump} = 2\pi \int_0^R r\, dr = \pi R^2$$

which coincide with the geometric derivation.

Example 3.2 The Total Variation approximating function can be discontinuous if the approximated function $h(x)$ is discontinuous. The discontinuity of $u(x)$ inside its interval of total variation does not affect the value of the total variation.

3.2 Continuity of the solution and the growth conditions

Superlinear growth Using similar arguments, we can show that a jump of an extremal for a Lagrangian of superlinear growth

$$\lim_{|u|' \to \infty} \frac{L(x, u, u')}{u'} = \infty$$

is not optimal, because the cost of any jump is infinite; hence, discontinues functions never occur in minimizing sequences: The penalty for discountinuity is infinitely high.

Therefore, the derivative $u'(x)$ in the optimal solution is uniformly bounded at any bounded interval $[a, b]$. It follows that all solutions of a variational problem with the Lagrangian of superlinear growth are bounded too if at least one boundary condition prescribes a finite the value of u: $u(a) = u_a$.

Example 3.3 The problems of Lagrange mechanics do satisfy this assumption because kinetic energy depends on the speed $\dot{x}$ quadratically. This corresponds to the physics: trajectories of inertial particles are continuous.

The approximations with a quadratic penalty are also continuous even if the approximated function is not.

Sublinear growth The Lagrangians of sublinear growth

$$\lim_{|u|' \to \infty} \frac{L(x, u, u')}{u'} = 0$$

admit any number of jumps, because the contribution of each jump to the cost functional is zero. The discontinuities in minimizing sequences are not penalized at all.

Sublinear growth at a point In the first example, The Lagrangian $x^2 u'^2$ grows superlinearly except at the point $x = 0$, where it grows sublinearly.

$$\lim_{|u'| \to \infty} \frac{L}{|u'|} = \begin{cases} \infty, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The discontinuity of $u(x)$ at $x = 0$ is not penalized.

Sublinear growth at several points Consider the problem with Lagrangian

$$L = (x - \frac{1}{2})^2 (x + \frac{1}{2})^2 u'^2$$

on the same interval $[-1, 1]$ and with the same boundary conditions: $u(-1) = -1, u(1) = 1$; L grows sublinearly at the points $x = \pm\frac{1}{2}$. Using the same arguments, we find the solution

$$u(x) = \begin{cases} -1, \text{ if } & -1 < x, -\frac{1}{2} \\ c, \text{ if } & -\frac{1}{2} < x < \frac{1}{2} \\ 1 \text{ if } & \frac{1}{2} < x < 1 \end{cases}$$

where c is an arbitrary constant. The solution is not unique. In the same way, one can construct the problem with the solution that has any number of jumps with undefined magnitudes of the jumps.

Sublinear growth in an interval The solution may have an everywhere dense set of jumps. This disconnects the function and its derivative

$$u(x) \to f(x), \; u'(x) \to g(x)$$

Indeed, consider a sequence of points $a = z_0, z_1, \dots z_N = b$, and assume that the distance between them less then $\epsilon_N > 0$. Set

$$u(z_n) = f_n$$

and define the piece-wise affine functions

$$\hat{u}_n(x) = u(x_n) + g_n(x - x_n), \quad x \in [x_n, x_{n+1}) \tag{12}$$

where $\{g_n\}$ is an arbitrary sequence. Sequence $\{\hat{u}\}(x)$ approximates $u(x)$ up to the order of ϵ_N:

$$|u(x) - \hat{u}(x)| = o(\epsilon_N) \quad \forall x \in [a, b]$$

The derivative $\hat{u}_n(x)$ is piece-wise constant

$$\hat{u}'_n(x) = g_i, \quad x \in [x_n, x_{n+1}) \tag{13}$$

at each interval $[x_n, x_{n+1})$ (notice that $g_i = \hat{u}'_n(x) \neq u'(x_n)$). At the end of each interval, $\hat{u}_n$ is discontinuous,

$$\begin{aligned} \lim_{x \to x_{n+1}-0} \hat{u}_n &= u(x_n) + g_i(x_{n+1} - x_n) \\ \lim_{x \to x_{n+1}+0} \hat{u}_n &= u(x_{n+1}) \end{aligned}$$

and $\hat{u}'_n(x)$ is undetermined. Because of sublinear growth of the Lagrangian, discontinuities of $\hat{u}$ do not affect the cost functional.

The variational functional, computed for the sequence $\hat{u}_n(x)$, becomes

$$\begin{aligned} I_N &= \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} F(x, \hat{u}_n(x), g_n) dx \\ &= \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} F(x, u_n, g_n) dx + O(\epsilon_N) \end{aligned}$$

The limit of such sequence when $N \to \infty, \epsilon_N \to 0$, depends on two independent functions $u(x)$ and $g(x)$, the variational problem becomes the problem of minimization with respect to these uncorrelated functions:

$$\lim_{N \to \infty} I_N = I = \min_{u(x)} \min_{g(x)} \int_a^b F(x, u(x), g(x)) dx$$

In particular, boundary conditions on u do not affect the cost because they can be satisfied by a discontinuity at the end of the interval $[a, b]$.

Example 3.4 Let F be

$$F = \sqrt{|u'|} + (u - x)^2$$

The minimizing sequence consists of the piece-wise constant step functions

$$\hat{u}_n(x) = u(x_n), \quad x \in [x_n, x_{n+1})$$

The derivative $\hat{u}_n'$ is zero everywhere except $x = x_n, n = 1, 2, ..$ where $\hat{u}_n$ is discontinuous. The discontinuities do not affect the Lagrangian.

The limiting Lagrangian F_0 is

$$F_0(u(x), g(x)) = \sqrt{|g(x)|} + (u(x) - x)^2,$$

the minimizer is

$$v = 0, u = x, \quad \text{and} \quad \int_a^b F_0(0, x)dx = 0$$

In the classical sense, the solution to this problem does not exist. However, the discontinuous solution is a minimizer even if it does not belong to the set of differentiable functions.

4 Regularization and viscosity solutions

A slight perturbation of the problem (regularization) yields to the problem that has a classical solution and this solution may be close to the discontinuous solution of the original problem. This time, regularization is performed by adding to the Lagrangian a stabilizer which is a strictly convex function $\epsilon\rho(u')$ of superlinear growth.

4.1 Regularization of a finite-dimensional linear problem

As the most of variational methods, the regularization has a finite-dimensional analog. It is applicable to the minimization problem of a convex but not strongly convex function which may have infinitely many solutions. The idea of regularization is to slightly perturb the function by small but a strictly convex term; the perturbed problem has a unique solution to matter how small the perturbation is. The numerical advantage of the regularization is the convergence of minimizing sequences.

Let us illustrate ideas of regularization by studying a finite dimensional problem. Consider a linear system

$$Ax = b \tag{14}$$

where x is an unknown n-dimensional vector, A is a known square $n \times m$ matrix and b is a known m-dimensional vector.

We know from linear algebra that the Fredholm Alternative holds:

- If $\det A \neq 0$, the problem has a unique solution:

$$x = A^{-1}b \quad \text{if } \det A \neq 0 \tag{15}$$

- If $\det A = 0$ and $A^T b \neq 0$, the problem has no solutions.
- If $\det A = 0$ and $A^T b = 0$, the problem has infinitely many solutions.

Remark 4.1 In practice, we also deal with an additional difficulty: The determinant $\det A$ may be a "very small" number and one cannot be sure whether its value is a result of rounding of digits or it has a "physical meaning." In any case, the errors of using the formula (15) can be arbitrary large and the norm of the solution is not bounded.

To address this difficulties, it is helpful to restate linear problem (14) as an extremal problem. Minimize the square of norm of the difference between Ax and b

$$\min_{x \in R^n} (Ax - b)^T (Ax - b) \tag{16}$$

This problem does have at least one solution, no matter what the matrix A is. We compute:

$$\frac{d}{dX}(Ax - b)^T (Ax - b) = 2[(A^T A)x - A^T b] = 0$$

The square $n \times n$ matrix $A^T A$ is non-negatively defined. If the inverse $(A^T A)^{-1}$ exists, we obtain

$$x = (A^T A)^{-1} A^T b$$

If $m = n$ and matrix A is non-singular, the last formula becomes

$$x = (A^T A)^{-1} A^T b = A^{-1} (A^T)^{-1} A^T b = A^{-1} b$$

This solution coincides with the solution of the original problem (14) when this problem has a unique solution; in this case the cost of the minimization problem (16) is zero. Otherwise, the minimization problem provides "the best approximation" of the non-existing solution.

If the problem (14) has infinitely many solutions, so does problem (16). Corresponding minimizing sequences $\{x^s\}$ can be unbounded, $\|x^s\| \to \infty$ when $s \to \infty$.

In this case, we may select a solution with minimal norm. We use the *regularization*, passing to the perturbed problem

$$\min_{x \in R^n} (Ax - b)^T (Ax - b) + \epsilon x^T x, \quad \epsilon > 0$$

The solution of the last problem exists and is unique. Indeed, we have by differentiation

$$(A^T A + \epsilon I)x - A^T b = 0$$

and

$$x = (A^T A + \epsilon I)^{-1} A^T b$$

We mention that

1. The inverse exists since the symmetric matrix $A^T A$ is nonnegative defined, and ϵ is positive. The eigenvalues $(A^T A + \epsilon I)$ are real numbers not smaller than ϵ.

2. Suppose that we are dealing with a well-posed problem (14), that is the matrix A is not degenerate. When $\epsilon \to 0$, the solution becomes the solution (15) of the unperturbed problem, $x \to A^{-1}b$.

3. If the problem (14) is ill-posed, the norm of the solution of the perturbed problem is still bounded:
$$\|x\| \leq \frac{1}{\epsilon}\|b\|$$
because eigenvalues of $(A^T A + \epsilon I)^{-1}$ are not greater than $\frac{1}{\epsilon}$.

Remark 4.2 Instead of the regularizing term ϵx^2, we may use any positively define quadratic $\epsilon(x^T P x + p^T x)$ where matrix P is positively defined, $x^T P x > 0\ \forall x$, or other strongly convex function of x.

4.2 Regularization of the example H

Similarly to the finite-dimensional problem, we perturb the Lagrangian $x^2 u'^2$ in the Example H (1) by adding a uniformly convex stabilizer $\epsilon^2 u'^2$ ($\epsilon > 0$). The perturbed problem for the Example H (1) becomes:

$$I_\epsilon = \min_{u(x)} I_\epsilon(u), \quad I_\epsilon(u) = \int_{-1}^{1} \left(x^2 u'^2 + \epsilon^2 u'^2\right) dx, \quad u(-1) = -1, \quad u(1) = 1, \tag{17}$$

After the perturbation $\epsilon^2 u'^2$ is added to the original Lagrangian $x^2 u'^2$, the perturbed Lagrangian is of superlinear growth everywhere.

The first integral of the Euler equation for the perturbed problem becomes

$$(x^2 + \epsilon^2)u' = C, \quad \text{or} \quad du = C\frac{dx}{x^2 + \epsilon^2}$$

Integrating and accounting for the boundary conditions, we obtain

$$u_\epsilon(x) = A \arctan\frac{x}{\epsilon}, \quad A = \left(\arctan\frac{1}{\epsilon}\right)^{-1}$$

When $\epsilon \to 0$, the solution $u_\epsilon(x)$ converges to the discontinuous minimizer $u(x)$ although the convergence is not uniform at $x = 0$.

We also show that the sequence of perturbed functionals $I(u_\epsilon)$ tends to the limiting value $I(u) = 0$. Indeed,

$$[u_\epsilon]' = A\frac{\epsilon}{\epsilon^2 + x^2};$$

therefore, the cost of the perturbed problem is of the order of ϵ^2, $I(u_\epsilon) = O(\epsilon^2)$. This proofs that the sequence of solutions $u_\epsilon(x)$ is a minimizing sequence.

4.3 Regularization of the minimal surface of revolution problem

Regularization by adding a pointwise constraint The Goldschmidt solution consists of two essentially disconnected circles; it shows that the connectivity is not a valid constraint in the problem of the minimal surface. A number of infinitesimal tunnels can be added to the surface; they changes the connectivity but do not increase the area.

A possible regularization of the problem is performed by adding a constraint. For example, we may require that the radius of this path is uniformly bounded from zero by a positive constant r_0,

$$r(z) \geq r_0 \quad \forall z \in [a, b].$$

This requirement says that a ball of the radius r_0 can pass through the tunnel. With such constraint, the solution splits into a cylinder of the radius r_0, $r(z) = r_0$ and the catenoid that satisfies the Euler equation and joins the cylinder with the boundary circle.

$$r(z) = \begin{cases} r_0 & \text{if } 0 \leq x \leq s \\ \frac{\cosh(C(z-C_1))}{C} & \text{if } s \leq z \leq a \end{cases}$$

The three constants s and C and C_1 are bounded by two constraints, that express the boundary condition and continuity of $r(x)$ at the point $z = s$ where the cylinder meets catenoid:

$$(C, C_1, s) := \left\{ R = \frac{\cosh(C(a - C_1))}{C}, \quad r_0 = \frac{\cosh(C(s - C_1))}{C} \right\}. \tag{18}$$

The area $A_{cat}(s, C, C_1)$ is given by (13). The minimal area is given by solution of the optimization problem

$$A(s, C, C_1) = 2 \min_{s, C, C_1 \text{as in (18)}} (2\pi r_0 s + A_{cat}(s, C, C_1))$$

It remains to find (numerically) optimal values s and C and C_1 that are subject to above constraints.

Regularization by adding an integral constraint Another way to regularize the problem is the adding an intergal constraint. For example, one can request that the volume under inside of the body bounded by the minimal surface and two circles is given

$$V = \pi \int_0^R r(z)^2 dz$$

Lagrangian becomes

$$F = r\sqrt{1 + r'^2} + \lambda r(z)^2 dz$$

where λ is the Lagrange multiplier.

Proceeding as before, we find

$$u' \frac{\partial F}{\partial u'} - F = \frac{r}{\sqrt{1 + r'^2}} - \lambda r(z)^2 - C = 0;$$

As before, we solve for r' and separate the variables; the solution in quadratures is

$$z = \int_0^r \sqrt{R(r)}\, dr, \quad R(r) = \frac{r}{2} \left(\frac{1}{\lambda r^2 + C - r} + \frac{1}{\lambda r^2 + C - r} - 1 \right)$$

Regularization by perturbation of the Lagrangian A common way for regularization is the replacement of the Lagrangian $F(u, u')$ with a perturbed one $F_\epsilon(u, u')$ so that

$$|F_\epsilon(u, u') - F(u, u')| \to 0, \quad \text{when } \epsilon \to 0$$

and the variational problem with Lagrangian $F_\epsilon(u, u')$ has a solution. The perturbed problem may not have a clear geometric sense.

Lagrangian of the minimal surface problem

$$F(r, r') = r\sqrt{1 + r'^2}$$

can be perturbed so that r is always separated from zero. For example, consider the perturbed Lagrangian in which we replace r with $\sqrt{r^2 + \epsilon^2}$, so that this term cannot do to zero:

$$F_\epsilon(r, r') = \sqrt{r^2 + \epsilon^2}\sqrt{1 + r'^2} \tag{19}$$

The difference $\Delta = |F_\epsilon(r, r') - F(r, r')|$ is equal to

$$\Delta = \epsilon^2 \frac{\sqrt{1 + r'^2}}{\sqrt{r^2 + \epsilon^2} + r}$$

and goes to zero when $\epsilon \to 0$.

The first integral of the perturbed Lagrangian is

$$r' \frac{\partial F}{\partial r'} - F = \sqrt{\frac{r^2 + \epsilon^2}{1 + r'^2}} = c \tag{20}$$

The left-hand side is larger that ϵ^2, therefore $c > \epsilon^2$ too. This time, r' in denominator is is bounded when c is bounded..

Problem: Derive the equation for the minimizer of Lagrangian (19), check its validity for all values of parameters.