Hamiltonian and Duality

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1 Hamiltonian and differential constraints

1.1 Differential constraints in Variational problems

Here, we derive Hamiltonian with all its remarkable properties from the analysis of a constrained variational problem.

The problem

$$I = \min_u \int_a^b F(x, u, u')dx$$

for the vector-valued minimizer $u(x)$ can be presented as the following constrained problem

$$I = \min_{u,v} \int_a^b F(x, u, v)dx \quad \text{subject to } u' = v \tag{1}$$

where the constraint $u' = v$ specifies the differential dependence between two arguments of the Lagrangian. The last problem is naturally rewritten using Lagrange function $p = p(x)$:

$$I = \min_{u,v} \max_p J, \quad J = \int_a^b \left[F(x, u, v) + p(u' - v)\right] dx \tag{2}$$

Integration by parts of the term $p\,u'$ in the integrand gives

$$J = \int_a^b F_D(x,u,v,p)dx + p\,u|_b^a \quad F_D(x,u,v,p) = [F(x,u,v) - p'u - p\,v]$$

Here, F_D is the dual form of the Lagrangian as a function of x, u, v, and p. Interchange the sequence of extremal operations in (2) and obtain the inequality:

$$I \geq \max_p I^D; \tag{3}$$

$$I^D = \min_{u,v} \int_a^b F_D(x,u,v,p)dx + p\,u|_a^b \tag{4}$$

Notice that the integrand for I^D includes u and v but not their derivatives, therefore the minimization is performed independently in each point of the trajectory The stationarity conditions for I^D are the coefficients by variations δu and δv, respectively

$$\frac{\partial F}{\partial u} - p' = 0, \quad \frac{\partial F}{\partial v} - p = 0. \tag{5}$$

Two equations (5) depend on four variables: u, v, p, and p'.

The boundary conditions are found form the variation of the term $p\,u|_a^b$: $p(b)\delta u(b) - p(a)\delta u(a) = 0$. If the value of u is not prescribed at the boundary point then $p = 0$ at this point. Now, we transform the problem in three different ways.

1.2 Primary and Dual Lagrangians, Hamiltonian

Excluding Lagrange function p. Original Euler equation We may exclude Lagrange function p and its derivative p' from (5). Differentiate second equation in (5) and subtract the result from the first equation; obtain

$$\frac{d}{dx}\frac{\partial F}{\partial v} - \frac{\partial F}{\partial u} = 0, \quad u' = v$$

Thus, we return to the Euler equation in the original form. This procedure is similar to excluding Lagrange multipliers in the finite-dimensional optimization problem.

Excluding the minimizer: Dual problem. Lower bound We may exclude u and v from two equations (5) solving them for u and v:

$$u = \phi(p,p'), \quad v = \psi(p,p') \tag{6}$$

Since $v = u'$, we obtain the dual from of stationarity condition

$$\frac{d}{dx}\phi(p,p') - \psi(p,p') = 0$$

that is a second-order differential equation for p.

The corresponding dual Lagrangian in p, p' variables is

$$F^D(p, p') = F(x, \phi, \psi) - p'\phi - p\psi \tag{7}$$

and the dual variational problem has the form

$$I \geq I^D; \quad I^D = \max_p \int_a^b \left[F^D(x, p, p')\right] dx \tag{8}$$

The dual problem (8) asks for maximum of the functional with Lagrangian $F^D(p, p')$, see (7). Any trial function p that is consistent with the boundary conditions produces the lower bound of I.

Remark 1.1 (Lower bound) Duality is an essential tool because it provides the estimate for a lower bound of the cost of variational problem. The upper bound of a minimization problem is easy to obtain: every trial function u_{tr} consistent with the main boundary conditions provides an upper bound for a minimal variational problem.

To find the lower bound, we consider the dual problem (8). Because (8) is a maximization problem, any trial function p_{tr} consistent with the main boundary conditions corresponds to the lower bound of the functional I^D and therefore for I:

$$\int_a^b F(x, u_{\mathrm{tr}}, u'_{\mathrm{tr}})dx \geq I \geq I_D \geq \int_a^b F^D(x, p_{\mathrm{tr}}, p'_{\mathrm{tr}})dx \quad \forall p_{\mathrm{tr}}, u_{\mathrm{tr}} \tag{9}$$

The difference between the upper and lower bound provides a measure of the accuracy of both approximations.

Excluding derivatives: Hamiltonian. canonical system Excluding v from (5): $v = \zeta(u, p)$ we express the problem through the Hamiltonian $H(u, p)$

$$I^H = \min_u \max_p \int_a^b \left[u'p - H(u, p)\right] dx \tag{10}$$

where

$$H(u, p) = \zeta(u, p)p - F(x, u, \zeta(u, p)) \tag{11}$$

Necessary conditions for maximum in (10) with respect to p recover the first condition in the system

$$u' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial u}.$$

To obtain the second condition, we first integrate by parts the term $u'p$ in the integrand in (10) replacing it with $-u\,p'$, then write the necessary conditions for minimum with respect to u. This argument explains the remarkable feature of Hamiltonian as the potential for the canonical system; this also clarifies the minus sign in the canonical equations.

2 Duality

2.1 Legendre transform

Legendre transform Duality in the calculus of variation is closely related to the duality in the theory of convex function; both use the same algebraic means to pass to the dual representation. Here we review the Legendre transform that define the dual Lagrangian.

Assume that $L(z)$ is a convex and twice differentiable function of a vector z. The conjugate (dual) function $L^*(p)$ is defined by the relation called Legendre transform of $L(z)$

$$L^*(p) = \max_z \left[p\, z - L(z) \right], \tag{12}$$

$L^*(p)$ is called the convex conjugate of a function $L(z)$ or dual to $L(z)$. It is also known as Legendre-Fenchel transformation or Fenchel transformation of $L(z)$.

Notice that maximum in the right-hand side of (12) exists because $L(z)$ is convex, and the maximum is unique if $L(z)$ is strictly convex. Computing the maximum, we write the stationary condition:

$$\frac{\partial}{\partial z} \left(p\, z - L(z) \right) = 0$$

or

$$p = L'(z), \tag{13}$$

Solve this equation for z,

$$z = \phi(p) \tag{14}$$

where $\phi = (L')^{-1}$ is an inverse function of L', and compute the dual of $L(z)$

$$L^*(p) = p\, z - L(z) = p\, \phi(p) - L(\phi(p)) \tag{15}$$

Hamiltonian is a conjugate to Lagrangian Consider the Lagrangian $F(x, u, u')$ as an algebraic function of the third argument u' with fixed x, u, $F(x, u, u') = L(u')$. The Legendre transform of $L(u')$ is identical to (15) if the argument u' is called z. Assume that the Lagrangian $H(x, u, u')$ is convex with respect to u' and compute the Legendre transform:

$$\max_{u'} \left\{ p\, u' - H(x, u, u') \right\} \tag{16}$$

An optimal u' satisfies the equation:

$$p = \frac{\partial H}{\partial u'} \tag{17}$$

which says that impulse p is dual to u', see (13). Equation (17) is solvable for u', because $F(., ., u')$ is convex. The Hamiltonian $H(x, u, p)$ turns out to be the dual function to the Lagrangian $L(x, u, u')$ with respect to the third argument u'.

Dual problem Observe that the dual Lagrangian $F^D(x,p,p')$ is also the Legendre transform of Lagrangian, but the transform is performed with respect to both arguments u and u'

$$F^D(x,p,p') = \max_{u,v} [[u,v] \cdot [p',p] - F(u,v)] \tag{18}$$

In this procedure, we view arguments p and p' as independent variables. Notice that p' is the dual to u variable.

The dual form of the Lagrangian can also be obtained from the Hamiltonian when the variable u is expressed as a function of p and p' and excluded from the Hamiltonian.

$$F^D(x,p,p') = \max_{u} [u\, p' - H(u,p)] \tag{19}$$

The dual equations for the extremal can be obtained from the canonical system if it is solved for u and becomes a system of n second-order differential equations for p.

2.2 Properties of Legendre transform

Here we list several useful properties of the Legendre transform.

The Legendre transform of a convex function is convex. Assume that $L(z)$ is a strictly convex twice differentiable function, that is $L''(z) > 0$ for all z. Equatons (13) and (14) imply that

$$L'(\phi(p)) = p$$

Also by theorem of inverse function

$$\frac{d\phi(p)}{dp} = \frac{1}{L''(\phi(p)}.$$

Therefore,

$$L^*(p) = p\,\phi(p) - L(\phi(p))$$

is a composition of differentiable functions. We compute, using product role

$$\frac{d(L^*)}{dp} = \phi(p) + (p - L'(\phi(p))) \cdot \frac{d\phi(p)}{dp} = \phi(p),$$

and then

$$\frac{d^2(L^*)}{dp^2} = \frac{d\phi(p)}{dp} = \frac{1}{L''(\phi(p))} > 0,$$

so $L^*(p)$ is convex.

Moreover, the second derivatives of a function and of its transformation satisfy the remarkable equality

$$\frac{d^2L^*(p)}{dp^2}\, \frac{d^2L(z)}{dz^2}\bigg|_{z=\phi(p)} = 1 \tag{20}$$

In particular, this implies that their first derivatives are orthogonal.

Geometric interpretation Consider graph of a convex function $y = L(z)$. Draw a straight line $p\,z + b$ below the groph and move move the line with the fixed slop p parallel to itself increasing b until the line touches the graph $L(z)$. When the line touches the graph, register the tangent p of its angle and the coordinate b of the intersection of the line with the axis OY, obtain $b(p)$. Then change the slop p and repeat the experiment; do it for all angles that is for all $p \in R$. If for some p every line intersects the graph, set $b = -\infty$.

The equation of the tangent line in the plane (z, y) at the point $z = z_0$ is

$$y - L(z_0) = p(z - z_0), \quad p = L'(z_0)$$

where z_0 is the intersection point. The tangent line intersects the the vertical axis $z = 0$ at the point

$$y = [L(z_0) - p\,z_0]$$

Since the line was moved parallel to itself until it first meets the graph of $L(z)$ at the point z_0, we write

$$y(z_0) = \min_{z_0} [L(z_0) - p\,z_0]$$

If we denote $L^* = -y$, change min and max accordingly, and omit index $_0$, we obtain the Legenbdre transform

$$L^*(z) = \max_z [p\,z - L(z)], \quad p = L'(z)$$

If $L(z)$ is strictly convex and twice differentiable function, the relation between $(z, L(z))$ and $(p, L*(p))$ is one-to-one mapping and $L(z)$ can be recovered back from its Legendre transform.

The Legendre transformation is an involution, i.e., $L^{**} = L$ The second conjugate or biconjugate (the convex conjugate of the convex conjugate) L^{**} to L or the conjugate to $L^*(p)$,

$$L^{**}(z) = \max_p \{p\,z - L^*(p)\}. \tag{21}$$

We denote the argument of L^{**} by t.

It is easy to show that the Legendre transformation is an involution, i.e., $L^{**}(z) = L(z)$. By using the above equalities for $\phi(z)$, $L^*(p)$ and its derivative, we find

$$(L^*)^*(t) = t\,p - L^* = t\,p - (t\,p - L) = L$$

Symmetric form of Legendre transform The inequality of the Legendre transform $L^*(p) \geq p\,L(z) \geq z\,p - L^*(p) \quad \forall z - L(z)$ implies a symmetric inequality (Fenchel's inequality)

$$p\,z \leq L(z) + L^*(p) \quad \forall z, p \tag{22}$$

This inequality can be applied to $L^*(p)$ istead of $L(z)$; it hints that the conjugate of $L^*(p)$ equals to $L(z)$, or $L^{**}(z) = L(z)$ for convex functions $L(z)$

Lower bound Duality can be used to estimate the minimum from below. The inequality (22) provides the lower estimate:

$$z, p$$

Choosing a trial value p we find the lower bound.

2.3 Examples

Example 2.1 (Find a dual) Assume that $L(z) = \frac{1}{a}z^a$ We compute

$$p = L'(z) = z^{a-1}, \quad z = \phi(p) = (p)^{\frac{1}{a-1}}, \quad L(z) = \frac{1}{a}p^{\frac{a}{a-1}}$$

and

$$L^*(p) = z\,p - L(z) = (p)^{\frac{a}{a-1}} - \frac{1}{a}p^{\frac{a}{a-1}} = \frac{1}{b}\,(p)^b$$

where $b = \frac{a}{a-1}$. Finally we write

$$\left(\frac{1}{a}z^a\right)^* = \frac{1}{b}z^b, \quad \text{where} \quad \frac{1}{a} + \frac{1}{b} = 1 \tag{23}$$

We observe that this transform is an involution. $L^{**}(z) = L(z)$

Special cases

Stable point If $a = \frac{1}{2}$, then $b = \frac{1}{2}$; the Legendre transform has a stable point:

$$L(z) = \frac{1}{2}z^2 \quad L^*(p) = \frac{1}{2}p^2.$$

Limiting case Assume that $a = 1 + \epsilon$, where $\epsilon > 0$ and $\epsilon \ll 1$. In this case $b \to 1 + \frac{1}{\epsilon}$.

$$L(z) = \frac{1}{1+\epsilon}|z|^{1+\epsilon}, \quad L^*(p) = \frac{\epsilon}{1+\epsilon}|p|^{\frac{1+\epsilon}{\epsilon}}.$$

In the limit $\epsilon \to 0$, we have

$$L(z) = |z|, \quad L^*(p) = p^\infty = \begin{cases} 0 & |z| < 1 \\ \infty & |z| > 1 \end{cases}.$$

The dual to $L(z) = |z|$ is the well function.

Geometrically, we find that any line with the slope $p \in (-1, 1)$ touches the graph $L(z) = |z|$ at the origin $(0, 0)$, therefore $L^*(p) = 0$ if $p \in (-1, 1)$. If $|p| > 1$, the touching point does not exist but if $\epsilon \to 0$, the touching point $p^{\frac{1}{\epsilon}}$ moves to infinity and the point of intersection of the tangent with OY-axis moves to $-\infty$, and we may say it is at an infinite point for the limiting case, therefore $L^*(p) = \infty$ if $p \notin [-1, 1]$

2.4 Further properties

The next translation and scaling properties of the transform are verified by a direct calculations. Let $m(z)$ be a twice differentiable function, and b, c, and k be some constants. We check that

$$L(z) = m(z) + b \Rightarrow L^\star(p) = m^\star(p) - b \tag{24}$$

$$L(z) = m(z + c) \Rightarrow L^\star(p) = m^\star(p) - p \cdot c \tag{25}$$

$$L(z) = k \cdot m(z) \Rightarrow L^\star(p) = k \cdot m^\star\left(\frac{p}{k}\right) \tag{26}$$

$$L(z) = m(k \cdot z) \Rightarrow L^\star(p) = m^\star\left(\frac{p}{k}\right). \tag{27}$$

The inversion property is:

$$L(z) = m^{-1}(z) \Rightarrow L^\star(p) = -p \cdot m^\star\left(\frac{1}{p}\right) \tag{28}$$

2.5 Extension of the class of transformed functions

1. The convex conjugate (dual) may be not defined for some values of p. In this case, we assume that it is equal to ∞ for these values

Example 2.2 (Find a convex conjugate) Consider

$$L(z) = \exp(z). \tag{29}$$

and find $L^*(p) = \max_z[p\, z - \exp(z)]$

We compute

$$\frac{dL(z)}{dz} = p - \exp(z) = 0, \quad z = \log(p),$$

The dual function is not defined for nonpositive p. It is geometrically obvious, that the line which touches the exponent must have a positive slop p. We define $L^*(p) = \infty$ if $p \leq 0$, and obtain

$$L^*(p) = \begin{cases} (p \log p - p) & \text{if} \quad p > 0, \\ \infty & \text{if} \quad p < 0. \end{cases} \tag{30}$$

2. The convex function $L(z)$ may be not differentiable at several vertex points z_i where $L''(z_i) \to \infty$. In this case, the dual function $L^*(z)$ is not strictly convex, the vertex points correspond to intervals of straight lines, where $L^{*\prime\prime}(z_i) \to 0$, see the duality relation (20).

Example 2.3 Consider

$$L(z) = \exp(|z|). \tag{31}$$

Using (32) we obtain

$$L^*(p) = \begin{cases} (p(\log|p| - 1) & \text{if} \quad |p| > 1, \\ 0 & \text{if} \quad |p| < 1. \end{cases} \tag{32}$$

Indeed, assume that $z > 0$. The slope $p = L'(z)$ is bigger than one, $p \geq 1$. then, $L(z)$ even function of z, its derivative is an odd function, therefore the corresponding slope $p = L'(z) \leq -1$ if $z < 0$. At $z = 0$, $L(z)$ has a vertex, the derivative jumps from -1 to 1. This point corresponds to an interval $L^*(p) = 0$ if $p \in (-1, 1)$.

3. The inverse statement is also true: If convex function $L(z)$ is not strongly convex and $L''(z) = 0$ at some intervals $z \in [z_{il}, z_{ir}]$, then the dual function $L^*(p)$ is not differentiable at several vertex points p_i where $L^{*\prime\prime}(p_i) \to \infty$.

Example 2.4 Find the dual to $L(z)$

$$L(z) = \begin{cases} \frac{1}{2}(|z| - 1)^2 & \text{if} \quad |z| \geq 1 \\ 0 & \text{if} \quad |z| \leq 1 \end{cases}. \tag{33}$$

We have

$$L^*(p) = \frac{1}{2}(|p| + 1)^2 - \frac{1}{2} \tag{34}$$

4. Legendre transform of function of linear growth

If $\lim_{z\to\infty} \frac{L(z)}{z} \leq q_+$, than $L^*(p) = \infty$ when $p > q_+$. Similarly, if $\lim_{z\to-\infty} \frac{L(z)}{z} \leq q_-$, than $L^*(p) = \infty$ when $p < -q_-$.

For example, the dual to the problem of geometric optics is improper for large p.

Example 2.5 Let $L(z)$ be

$$L(z) = w\sqrt{1 + z^2}, \quad w > 0$$

It grows linearly:

$$\lim_{|z|\to\infty} \frac{L(z)}{z} = w$$

Compute $Z^*(p)$. We have

$$p = \frac{dL}{dz} = \frac{z\,w}{\sqrt{1 + z^2}}, \quad z = \frac{p}{\sqrt{w^2 - p^2}}, \quad L(z(p)) = \frac{w^2}{\sqrt{w^2 - p^2}}$$

and

$$Z^*(p) = p\,z(p) - L(z(p)) = \begin{cases} -\sqrt{w^2 - p^2}, & |p| < w \\ \infty. & |p| \geq w \end{cases}$$

The slop of the touching line must be smaller than w and larger than $-w$.

For any Lagrangian that grows not slower than an affine function:

$$L(z) \geq c_1 + c_2\|z\| \quad \forall z, \tag{35}$$

where c_1 and c_2 are constants, $c_2 > 0$ $L * (p)$ is finite at least at some intervals of p.

Legendre-Fenchel transform of non-convex function Is is geometrically clear that the Legendre transform can be defined also for a larger class of non-differentiable functions. The generalization is called Legendre-Fenchel transformation.

Assume that $L(z)$ is not convex if $z \in (a, b)$. It is geometrically obvious, that $L^*(p)$ does not depend on the behavior of $L(z)$ in an interval of non-convexity. Particularly, dual $L^*(p)$ stays the same if $L(z)$ is replaced with its the convex envelope $\mathcal{C}L(z)$:

$$L^*(p) = (\mathcal{C}L)^*(p)$$

Since $\mathcal{C}L(z)$ is convex, the second conjugate coincides with it:

$$\mathcal{C}L^{**}(z) = \mathcal{C}L(z). \tag{36}$$

Because the conjugate $L^*(p)$ is the same for $L(z)$ and its convex envelope $\mathcal{C}L(z)$, the second conjugate to a non-convex function $L(z)$ is its convex envelope:

$$L^{**}(z) = \mathcal{C}L(z)$$

This property of second conjugate might be used to calculate the convex envelope.

2.6 Legendre transform of a function of several variables

Legendre transform of a function of a vector argument $z \in R^n$ is defined as respect to a straight generalization of one-variable case

$$L^*(p) = \max_z [p^T z - L(z)]$$

where $p \in R^n$ is the vector of conjugate variables.

Geometrical sense of the transform is similar to the one-variable case. For example, Legendre transform of function of two variables Legendre transform with respect to both variables corresponds to the following procedure. The graph of the function $L(z)$ in the three-dimensional space is touched by a plane $l = p_1 z_1 + p_1 z_2 + b$ fixed normal $n = (n_1, n_2)$ from below. The normal correspond to the gradient $n = \nabla l = [p1, p-2]$. Parameter $b(p)$ records the position of the touching plane; it is equal to the coordinate of intersection of the plane l with the axis OZ. The procedure is repeated for all normals n that is for all vectors p.

For convex functions, the transform is an involution, $L^{**}(z) = L(z)$, for non-convex functions, the second dual returns the convex envelope of the function, $L^{**}(z) = \mathcal{C}L(z)$.

Example 2.6 The dual to the function

$$L(z_1, z_2) = \frac{1}{2} z_1^2 + \frac{1}{\alpha} |z_2|^\alpha, \quad \alpha > 1$$

is

$$L^*(p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{m}|p_2|^\beta, \quad \beta = \frac{\alpha}{\alpha - 1}$$

In the limit, $\alpha \to 1$, we obtain

$$L(z_1, z_2) = \frac{1}{2}z_1^2 + |z_2|, \quad L^*(p_1, p_2) = \frac{1}{2}p_1^2 + \begin{cases} 0 & |p_2| < 1 \\ \infty & |p_2| > 1 \end{cases}$$

The dependence of p_2 in $L^*(p_1, p_2)$ is the index function of the set $p_2 \in (-1, 1)$

$$L^*(p_1, p_2) = \frac{1}{2}p_1^2 + \begin{cases} 0 & |p_2| < 1 \\ \infty & |p_2| > 1 \end{cases}$$

Legendre transform of a function of several variables may be performed with respect to any group of the variables. The other variables are considered as unvariable parameters.

Example 2.7 Consider a quadratic function of two arguments

$$L(z_1, z_2) = \frac{1}{2}\left(a\, z_1^2 + 2b z_1 z_2 + c z_2^2\right) = \frac{1}{2} w_{z_1, z_2}^T A_{z_1 z_2} w_{z_1, z_2}$$

where

$$w_{z_1 z_2} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad A_{z_1 z_2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

There are three conjugates that correspond to Legendre transform with respect to z_1 or to z_2, or to both. We will denote them as following

$$L^{*(z_1)}(p_1, z_2), \quad L^{*(z_1 z_2)}(p_1, p_2), \quad L^{*(z_2)}(z_1, p_2),$$

showing in the superscript the variables that are excluded by Legendre transform; the actual arguments are also listed in usual manner.

Compute these conjugates.

1. Compute $L^{*(z_1)}(p_1, z_2)$ We have

$$L^{*(z_1)}(p_1, z_2) = \max_{z_1}[p_1 z_1 - L(z_1, z_2)]$$

The optimal value of $\hat{z_1}$ of z_1 is found from the relation

$$p_1 = \frac{\partial}{\partial z_1} L(z_1, z_2) = a\hat{z_1} + b z_2$$

We obtain $\hat{z_1} = \frac{1}{a}(p_1 - b z_2)$; exclude $\hat{z_1}$ and find

$$L^{*(z_1)}(p_1, z_2) = p_1 \hat{z}_1 - L(\hat{z}_1, z_2) = \frac{1}{2} w_{p_1 z_2}^T A_{p_1 z_2} w_{p_1 z_2}$$

where

$$w_{p_1 z_2} = \begin{pmatrix} p_1 \\ z_2 \end{pmatrix}, \quad A_{p_1 z_2} = \frac{1}{a}\begin{pmatrix} 1 & b \\ b & b^2 - a\, c \end{pmatrix}$$

2. Compute $L^{*(z_2 p_1)}(z_1, p_2)$ The calculation is identical to the previous if we interchange indices 1 and 2 and permute entries in matrix $A_{z_1 p_2}$

$$L^{*(z_2)}(z_1, p_2) = \frac{1}{2} w_{z_1 p_2}^T A_{z_1 p_2} w_{z_1 p_2}$$

where

$$w_{z_1 p_2} = \begin{pmatrix} z_1 \\ p_2 \end{pmatrix}, \quad A_{z_1 p_2} = \frac{1}{c} \begin{pmatrix} b^2 - a\,c & b \\ b & 1 \end{pmatrix}$$

Notice that $\det A_{z_1 p_2} = -\frac{1}{a}$, $\det A_{p_1 z_2} = -\frac{1}{c}$, and $A_{p_1 z_2} A_{z_1 p_2} = \frac{1}{a\,c} Id$.

3. Compute $L^{*(z_1 z_2)}(p_1, p_2)$ Similarly, we find

$$L^{*(z_1 z_2)}(p_1, p_2) = \max_{z_1, z_2} [p_1 z_1 + p_2 z_2 - L(z_1, z_2)]$$

The optimal values $\hat{z}_1$ of z_1 and $\hat{z}_2$ of z_2 are found from

$$w_{p_1 p_2} = A_{z_1 z_2} w_{\hat{z}_1 \hat{z}_2}$$

and are

$$w_{\hat{z}_1 \hat{z}_2} = A_{z_1 z_2}^{-1} w_{p_1 p_2}$$

Excluding these variables, we obtain

$$L^{*(z_1 z_2)}(p_1, p_2) = \frac{1}{2} w_{p_1 p_2}^T A_{p_1 p_2} w_{p_1 p_2}$$

where

$$w_{p_1 p_2} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad A_{p_1 p_2} = A_{z_1 z_2}^{-1} = \frac{1}{a\,c - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

In thermodynamics, a similar transformation converts the internal energy to entalpy, then to Gibbs energy, Helmholtz energy, and back to the internal energy. The pairs of dual variables are temperature and entropy, and deformation and stress.