Introduction to Control Theory

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1 Optimal control

The theory of optimal control was developed starting from the1950s to meet the needs of designed automatic control systems. Optimal control problem is essentially the variational problem with some additional constraints.

1.1 Formulation

Preliminaries Optimal control theory was developed to solve the problem of optimal navigation of a mechanical system. It is assumed that the motion is described by a system of differential equations and is controlled by some parameters, called controls $u(t)$ which depend on time t. The equation of motion are written in a standard form which allow for the universal analysis of a large class of problems.

The motion of the system is described by an N dimensional vector $z(t) = [z_1(t), \dots z_N(t)]$ that is a solution to a system of differential equations in Cauchy form

$$z_i' = f_i(t, z, u), \quad z_i(0) = Z_i, \quad i = 1, \dots, N \tag{1}$$

with the known initial conditions $Z_1, \dots Z_N$. The right-hand side $f = (f_1, \dots f_N)$ is a bounded function that depends on independent variable t, on vector z and on a vector $u(t)$ of controls. The control $u(t)$ takes values in the known bounded set $\mathcal{U}$ of controls,

$$u \in \mathcal{U}(z, t) \tag{2}$$

in each point t. The control is allowed to instantly vary within $\mathcal{U}$ at any value of t. The set of controls may depend on time t and trajectory z.

For example, differential equation (1) may describe the motion of a car, and control $u(t) = (u_1(t), u_2(t), u_3(t))$ consists of a position u_1 of the gas pedal, angle of the steering wheel u_2, and position u_3 of the gear stick. Here u_1 and u_2 are real numbers that vary in a range $u_i \in (U_i^-, U_i^+)$ and u_3 is a discrete variable that takes several values. We assume that if the controls $u(t)$ and the initial conditions Z are known, one can find a differentiable solution $z(t)$ from (1).

Typically (but not always) the goal of the control is the maximization of a function $\phi(z)$ at the final time T. The optimization problem is formulated as:

$$I = \max_{u \in \mathcal{U}} \phi(z(T)) \quad \text{subject to } (1), (2) \tag{3}$$

The goal functional of the form (3) does not limit the generality of the formulation. We will show next how to transform a variational problem to this form.

A simplest variational problem as a control problem A classical variational problem

$$I = \min_w \int_0^T F(t, w, w')dt, \quad w(0) = w_0 \tag{4}$$

can be reformulated as an optimal control problem.

The minimizer w is a differentiable function, but its derivative w' is free of any point-wise constraints. We rewrite (4) introducing a new variable $u = z'$ called *control*. The control can be arbitrarily assigned at each point of trajectory. When the control is fixed, the trajectory w is uniquely defined. However, here the set of control is not bounded, $\mathcal{U} = R$.

We rewrite variational problem (4) in the form (1), (3). Variable w will be called z_1, it satisfies the equation $z_1' = u$. We also introduce a new variable:

$$z_2'(t) = F(t, z_1, u), \quad z_2(0) = 0$$

where z_2 equal to the accumulated value of the integral $z_2(x) = \int_0^x F(t, w, w')dt$. One can see from (4) that $z_2(T) = I$.

The problem (3) takes the form

$$I = \min_{u \in R} z_2(T) \quad \text{subject to} \tag{5}$$

$$z_1' = u, \quad z_2' = F(t, z_1, u), \quad z_1(0) = w_0, \quad z_2(0) = 0. \tag{6}$$

In the same fashion, we account for isoperimetric constraints

$$I = \int_0^T \Phi(t, w, w')dt.$$

We introduce a new variable z_3 equal to accumulated quantity:

$$z_3' = \Phi(t, z_1, u), \quad z_3(0) = 0, \quad z_3(T) = I.$$

Here, as before, $z_1 = w$ and $u = w'$. Variable z_3 satisfies two boundary conditions which is either possible by a special choice of control $u(t)$ or is impossible at all. Recall, that isoperimetric problems in calculus of variations have similar features. For example, in the problem of a hanging chain, the given length of the chain must be larger than the distance between the points of supports of its ends.

Remark 1.1 It is always possible but not necessary to transform integral goal functional into the optimization of a variable at the end point. We may as well optimize the classical integral functional

$$I = \min_w \int_0^T F(t, w, w')dt + b\, w(T) \tag{7}$$

and also keep isoperimetric constraints in their original form.

A generic form of a control problem Summary: In control problems, there are two groups of variables

1. Controls $u = [u_1 \dots, u_k]$ that are arbitrarily assigned at each point and may be subject to some point-wise algebraic constraints

$$u \in \mathcal{U}(z, t)$$

so that the controls u_i are bounded.

2. Phase variables $z = [z_1, \dots, z_n]$ that are (i) differentiable and (ii) entirely defined by the controls through differential constraints that are represented in the Cauchy form:

$$z_i' = f_i(t, z, u), \quad i = 1, \dots, N \tag{8}$$

Some boundary conditions are applied

$$b(z(0), z(T)) = 0 \tag{9}$$

where $b(a, b)$ is a give function of two vector arguments. The goal functional may consist of an integral term and a boundary term:

$$I = \int_{t_0}^{T} \Phi(t, z, u) dt + \phi(z(T)). \tag{10}$$

This formulation separates the control from the goal; they are connected via differential equations of motion.

1.2 Adjoint system

Consider the problem (10), (8), (9). Accounting for the differential constraints with Lagrange functions $\mu_i = \mu_i(t)$, the minimization problem can be rewritten in the form

$$\int_{t_0}^{t_1} L(t, z, z', u, \mu) dt + \phi(z(t_1)), \quad u(t) \in \mathcal{U} \tag{11}$$

where the Lagrangian $L(t, z, z', u, \mu)$ is

$$L(t, z, z', u, \mu) = \Phi(t, z, u) + \mu^T [-z' + f(t, z, u)]$$

The stationary conditions (Euler equation) with respect of z are:

$$\delta z_i : \left(\frac{\partial \Phi}{\partial z_i} + \mu^T \frac{\partial f}{\partial z_i} + \mu_i' \right) = 0, \quad i = 1, \dots, N \tag{12}$$

with the variational boundary conditions:

$$\left[\delta z_i \left(\mu_i\right)\right]_{t=t_0} = 0, \quad \left[\delta z_i \left(\mu_i + \frac{\partial \phi}{\partial z_i} \right)\right]_{t=t_1} = 0, \quad i = 1, \dots, N \tag{13}$$

Because of the assigned initial conditions, $\delta z_i(t_0) = 0$ and $\delta z_i(t_1)$ are arbitrary.

The Lagrange functions (dual variables) satisfy the system of linear differential equations that has a standard form:

$$\mu_i' = -\mu^T \frac{\partial f}{\partial z_i} - \frac{\partial \Phi}{\partial z_i} \tag{14}$$

$$\mu_i + \left.\frac{\partial \phi}{\partial z_i}\right|_{t=T} = 0 \tag{15}$$

Generally, the boundary conditions for μ are derived as stationarity conditions for the end point of the trajectory

$$\mu^T \cdot \delta z|_{t=0} = 0, \quad \left(\frac{\partial \phi}{\partial z} + \mu^T\right) \cdot \delta z|_{t=T} = 0 \tag{16}$$

where δz may be subject to constraints originated from the given initial or boundary conditions.

Notice that the total order of the system for differential constraints and Lagrange functions is $2N$, and there are $2N$ boundary conditions for it.

Hamiltonian The system (10), (8), (9) is further rewritten through Hamiltonian

$$H = z' \frac{\partial}{\partial z'} L(t, z, z', \mu, u) - L(t, z, z', \mu, u)$$

We compute

$$\frac{\partial L}{\partial z'} = -\mu, \quad z' = f(t, z, u)$$

and

$$H(z, \mu, u) = \mu^T f(t, z, u) - \Phi(t, z, u) \tag{17}$$

The stationarity equations take the canonical form

$$\mu' = \frac{\partial H}{\partial z} = -\mu^T \frac{\partial f}{\partial z}, \quad z' = -\frac{\partial H}{\partial \mu} = f(t, z, u) \tag{18}$$

The Hamiltonian contains all the information of the extremal problem and it is an algebraic function of its arguments.

1.3 Stationarity conditions for controls

It remains to find optimality conditions for optimal control. The specifics of control problems is that vector u of control varies in a bounded and may be multi-connected domain, $u \in \mathcal{U}$, and the controls can take values at the boundaries of $\mathcal{U}$.

First, we discuss a case when each entry of the control vector varies in an interval

$$u = (u_1, \dots, u_m), \quad u_i \in [U_i^-, U_i^+], \quad i = 1, \dots, m$$

The stationarity condition with respect to u_i is

$$\frac{\partial}{\partial u_i}\left[-\Phi(t,z,u)-\mu^T f(t,z,u)\right]\delta u_i = \frac{\partial H}{\partial u_i}\delta u_i \geq 0 \tag{19}$$

The optimal control may lie in the interior of the interval $[U_i^-, U_i^+]$ or at one of the boundaries; in the last case δu_i is either positive (if $u_i = U_i^-$) or negative (if $u_i = U_i^+$). We obtain the optimality relations

$$\frac{\partial H}{\partial u_i} = 0 \quad \text{if} \quad u_i \in (U_i^-, U_i^+) \tag{20}$$

$$\frac{\partial H}{\partial u_i} \geq 0 \quad \text{if} \quad u_i = U_i^- \tag{21}$$

$$\frac{\partial H}{\partial u_i} \leq 0 \quad \text{if} \quad u_i = U_i^+ \tag{22}$$

that consist of one equality of u_i and the inequalities for the interval where this equality holds.

Linear problem with quadratic functional As an example, consider the problem with a scalar control $u \in \mathcal{U}$, $\mathcal{U} = [a, b]$

$$\min_u \frac{1}{2}\int_{t_0}^{T}\left(c\,u^2\right)\,dt + B^T z|_{t=T} \tag{23}$$

$$z' = Az + Du, \quad z(t_0) = Z, \tag{24}$$

where A is an $N \times N$ matrix, D is an $N \times k$ matrix, and B are N-vector.

Hamiltonian is

$$H = -\frac{1}{2}c\,u^2 \;\; - \mu^T(Az + Du) \tag{25}$$

The dual system is

$$\mu' = \frac{\partial H}{\partial z} = -A^T\mu, \quad \mu(T) = B. \tag{26}$$

The stationarity conditions for the control are

$$\delta u \frac{\partial H}{\partial u} = \delta u(-c\,u - D^T\mu) \geq 0 \tag{27}$$

Let us call $u_s = -\frac{1}{c}D^T\mu$.
If $u_s \in (a, b)$, then optimal control u_0 equals u_s, $u_0(t) = -\frac{1}{c}D^T\mu(t)$; if $u_s < a$, then $u_0 = a$ and if $u_s > b$, then $u_0 = b$

When $c = 0$, the control takes only boundary values. This solution is called bang-bang control.

2 Pontryagin's maximum principle

2.1 Optimization problem

Needle variation Because of the constrained set of controls, the Weierstrass-type variation is not always possible. Indeed, this variation would require such a perturbation of u that its average value is zero, because u plays the role of derivative in the classical variation problem. If u is on the boundary (say in a corner point) of the set $\mathcal{U}$, this variation is not allowed.

Instead of the Weierstrass variation, we may use the needle-type variation (or McShane variation)

$$\Delta u(t) = 0, \text{ if } t \notin [t_*, t_* + \epsilon], \quad u(t) + \Delta u = U \in \mathcal{U}, \text{ if } t \in [t_*, t_* + \epsilon].$$

The control $u(t)$ changes its value to any point $u + \Delta u \in \mathcal{U}$ in the set of controls at the infinitely small interval $[t_*, t_* + \epsilon]$. This variation does not assume that the set $\mathcal{U}$ is connected; the set $\mathcal{U}$ may even consists of several isolated points. The only requirement is that the control switches its value in a small interval.

Unlike the Weierstrass variation, the needle variation does not imply that the variation of the trajectory δz is zero outside of the interval $[t_*, t_* + \epsilon]$ of variation; the variation is of the order of ϵ everywhere. The main term of the increment is of the order of ϵ and consists of two parts:

$$\delta I = \epsilon([H(t_*, ,., u, .) - H(t_*, .., U, .)] + \int_{t_0}^{T} \mu \left(\delta z' - \frac{\partial H}{\partial z}\delta z\right) dt + o(\epsilon)$$

The first term is the main term of the expansion of the integral

$$\int_{t_*}^{t_*+\epsilon} [H(t, z(u), \mu(u), u) - H(t, z(U), \mu(U), U)]dt =$$
$$\epsilon[H(t_*, z, \mu, u) - H(t_*, z, \mu, U)] + o(\epsilon).$$

Here, the continuity of z and μ is used. In the perturbed system, these quantities differ from the optimal values not more than by the term of the order of ϵ, see (1), (12), and therefore can be replaced with the optimal values in the approximation of the order of ϵ. The second term

$$\int_{t_*}^{T} \mu \left(\delta z' - \frac{\partial H}{\partial z}\delta z\right) dt = -\int_{t_*}^{T} \left(\mu' + \frac{\partial H}{\partial z}\right) \delta z \, dt + \mu' \delta z|_{t_0}^{T}$$

also contains terms of the order of ϵ. This term, however, vanishes due to the choice of the adjoint variables by virtue of (**??**; due to this choice, the variation of the Lagrangian with respect of δz is zero up to $O(\epsilon^2)$.

The remaining first term of the increment is nonnegative for all Δu such that $u + \Delta u \in \mathcal{U}$. In other words, optimal control u_{opt} delivers the minimum of the Hamiltonian:

$$u_{opt}(t) = \arg\left\{\min_{u \in \mathcal{U}} H(t, z, \mu, u)\right\} \quad \forall t \in (t_0, T) \tag{28}$$

where z and μ are computed along the optimal trajectory. This condition is called the *Pontryagin's maximum principle*

Remark 2.1 Traditionally, problem of the control theory is to maximize (not minimize) the functional; wherefore the name *maximum principle* is originated.

Now we have the complete set of equations to determine u, z, μ: N first-order differential equations (1) for the differential constraints, N first-order differential equations (12) for the adjoint system, and k equations (28) for an optimal control. The system is supplemented by $2N$ boundary conditions (16) and (27).

In summary, optimal control in each point of the trajectory is a solution to the constrained optimization problem (28); it depends on the function $H(u)$ and on the set of admissible controls $\mathcal{U}$. The optimal control may belong to the interior of $\mathcal{U}$ or to its boundary. In contrast, the classical variational problems deal with the minimum in an open set, which may lead to a solution with infinite control (discontinuous minimizer).

Control inside $\mathcal{U}$ Particularly, if u is inside $\mathcal{U}$, one can make $U - u$ infinitely small and obtain the stationary conditions

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial^2 H}{\partial u^2} \geq 0 \text{ if } u \in Int\,(\mathcal{U}) \tag{29}$$

These equations serve to find u.

Control on the boundary of $\mathcal{U}$ If u is on the boundary of $\mathcal{U}$, $u \in \partial\mathcal{U}$, the optimality condition depends on the type of the boundary point (a point on a smooth component of the boundary, a corner point, an isolated point, etc.) These conditions are differently expressed in these cases. For example, if the constraint on a scalar control restricts its values, $u_- \leq u(t) \leq u_+$ the conditions for the small variations are

$$\begin{aligned} u &= u_- \quad \text{if} \quad \frac{\partial H}{\partial u} > 0, \\ u &= u_+ \quad \text{if} \quad \frac{\partial H}{\partial u} < 0, \\ \frac{\partial H}{\partial u} &= 0, \quad \frac{\partial^2 H}{\partial u^2} \geq 0, \quad \text{if} \quad u_- \leq u(t) \leq u_+. \end{aligned}$$

In all cases, we have one equality to find the optimal control and one inequality to check. The maximum principle (28) is obviously stronger then these last conditions, but its verification is also more difficult.

2.2 Chattering regimes of control

The Hamiltonian $H(t, z, u)$ is generally nonconvex function of $u \in \mathcal{U}$. Moreover, the set $\mathcal{U}$ is generally also nonconvex. Therefore, minimizing H with respect of u we may find the global minimum u_0 and local minima $u_1, \dots u_m$. When $H(t,z,u)$ slowly changes together with $z(t)$ and t, one of the local minima, say u_1, may become global, then the global minimum u_0 becomes a local one. At an instance, both minima are equal, $H(t,z,u_0) = H(t,z,u_1)$. At this point, the optimal control may jump from the branch $u = u_0(z,t)$ to branch $u = u_1(z,t)$. Next, the optimal control may either follow the branch u_1, or it may jump back to u_0. It the last case, the control jumps again to u_1, then back, and these jumps come out infinitely fast. Such control regimes are called *chattering regimes of control*.

The homogenized (averaged, relaxed) description of the problem includes all possible fast-oscillating regimes. It is obtained by replacing the set $\{H(.,.,u), u \in \mathcal{CU}\}$ with its convex envelope that is the minimal convex set that contains $\mathcal{U}$. The added parts of the boundary of control set $\mathcal{CU}$ that do not coincide with the boundary of $\mathcal{U}$ correspond to infinitely fast oscillations of the controls.

The relaxed Hamiltonian $\mathcal{C}H$ is a convex function of u, and u belongs to the convexified set $\mathcal{C}U$. Therefore, the problem of minimization of $\mathcal{C}H(.,.,u)$ for $u \in \mathcal{C}U$ does not have local minima but only the global minimum.

Example 2.1 The control problem is

$$I = \inf J, \quad J = \int_0^{2\pi} [z - \cos(t)]^2(t) dt,$$

$$z(0) = .1, \quad z(2\pi) = -.1 \quad z' = u, \quad \mathcal{U} = \{1, \ -1\}$$

The admissible trajectories $z(t)$ are piece-wise straight lines with slopes equal to ± 1. The control function determines the instances of switching the slop.

We have

$$H(z,u) = -[z - \cos(t)]^2 - \mu u, \quad \mu' = -2(z - \cos(t))$$

and

$$u = arg \min\{H(z,u_1), H(z,u_2)\} = \begin{cases} 1, & \mu > 0 \\ -1 & \mu < 0 \end{cases}$$

where $u_1 = 1$ and $u_2 = -1$. The optimal control is $u = u_1$ if $z < \cos(t)$ and $u = u_2$ if $z > \cos(t)$. When $z = \cos(t)$, the control infinitely often oscillates between values u_1 and u_2 with time fractions m and $1 - m$. These fractions are found from the equation:

$$z' = \sin(t) = u_{hom} = m u_1 + (1 - m) u_2 = 2m - 1.$$

Here u_{hom} is the homogenized value of the control

$$u_{hom} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} u(x) dx$$

Finally, we find optimal time fraction

$$m(t) = \frac{\sin(t) + 1}{2}$$