Immediate Applications

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1 Introduction to Lagrangian mechanics

Leibnitz and Mautoperie suggested that any motion of a system of particles always minimizes a functional called action; later Lagrange came up with the exact definition of that action: the functional that has the Newtonian laws of motion as its Euler equation or stationarity condition. The question of whether the action reaches the actual minimum is complicated: Generally, it does not. We show below that the actual motion of particles reaches either a local minimum or a saddle point of action. The variational formulation permits a regular derivation of motion with Newtonian forces as an Euler equation of the action. The variational principles remain the abstract and economical way to describe Nature, but one should be careful in proclaiming the ultimate goal of the Universe.

1.1 Action Stationary Principle (Lagrange Principle)

Lagrange observed that the second Newton's law for the motion of a particle,

$$m\ddot{x} = f(x)$$

where $x(t)$ is the coordinate of the point, and t is time, and the dot stays for time derivative, can be viewed as the Euler equation for the variational problem:

$$A = \min_{x(t)} \int_{t_0}^{t_1} (T - V)\, dt, \tag{1}$$

$$T = \frac{1}{2} m (\dot{x})^2, \quad V = -\int f(x) dx, \tag{2}$$

Here t_0 is the time of the beginning of the motion, t_1 is the time of observation, A is the action,V is the potential energy, the negative of antiderivative of the force f, and $T(\dot{x})$ kinetic energy. Potential energy is negative to the work of the force f produced by the motion of a particle along the trajectory x

$$V = -\int f(x) dx$$

It is defined up to a constant.

Indeed, the Euler equation of A is:

$$m\ddot{x} + \frac{dV}{dx} = m\ddot{x} - f(x) = 0.$$

which shows that equation of Newton's law for a particle is the Euler equation for the stationarity of action A. One can check, the Legendre and Weierstrass tests are satisfied.

$$\frac{\partial^2}{\partial \dot{x}^2}(T - V) = m > 0$$

is satisfied.

Example 1.1 (Harmonic oscillator) The kinetic energy K of a mass m is proportional to the square of the speed $K = \frac{1}{2}m\dot{x}^2$, where m is the mass. The force f in the spring is proportional to its deflection x and is directed in the opposite direction of the deflection, $f = -kx$, where $k > 0$ is the stiffness of the spring. The the potential energy is $V = \frac{1}{2}kx^2$ Notice that the energy V is the convex function of x. The action is

$$A = \int_{t_0}^{t_f} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right) dx$$

The Euler equation $m\ddot{x} + kx = 0$ is the equation of the harmonic oscillator.

The same principle works for the system of several degrees of freedom.

Example 1.2 (Motion of a point in a constant gravitational field) Consider a free particle moving in the constant gravitational field, call the coordinates of the particle x, y, z, and assume that the axes z is directed vertically up.

The gravitational force $f = -mg$ is proportional to the mass m of the particle, g is the gravitational constant. The potential energy V of the gravitational force is $V = mgz$. The kinetic energy T of the mass is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The action is

$$A = \int_{t_0}^{t_1} \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{y}^2) - mgz \right) dx$$

There are three minimizers $x(t), y(t)$ and $z(t)$. The Euler equations for them are

$$m\ddot{x} = 0, \quad m\ddot{y} = 0, \quad m\ddot{z} + mg = 0$$

The equation of the motion are

$$x(t) = x_0 + v_x t, \quad y(t) = y_0 + v_y t, \quad z(t) = z_0 + v_z t - \frac{1}{2}mgt^2$$

where x_0, y_0, and z_0 are coordinated of the initial position of the particle, and v_x, x_y, and v_z are components of the initial speed.

As we have seen at the above examples, action $L = T - V$ does not satisfy Jacobi condition because kinetic and potential energies, which are both convex functions or $\dot{x}(t)$ and $x(t)$, enter the integrand for action with different signs. Generally, the action is a saddle function of $x(t)$ and $\dot{x}(t)$. The notion that Newtonial mechanics is not equivalent to minimization of a universal quantity had significant philosophical implications; it destroyed the hypothesis of universal optimality of the world. We should mention, however, that Newton's equation deals with the balance of instant quantities, acceleration and force, in each moment of time. This balance does not implies the minimization of any time integral quantity.

We could try to solve an inverse problem: Find a variational minimization problem solution of which provides the Newtonian equations as the Euler equations. Several minimal variational principles have been suggested by Gauss, Rayleigh, Hertz, and other great mathematicians. We do not discuss them here but refer to *Lanczos, Cornelius (1970). The Variational Principles of Mechanics (4th ed.). New York: Dover Publications Inc.*

1.2 Relativistic approach

Analyzing the form of the Lagrangian: $L = W - T$ one may ask: Why does the $(-)$ sign appear in the formula? It doesn't seem logical to subtract one form of energy from the other one. Surprisingly, the answer could be given by the Einstein's special relativity!

Consider maximization of the energy $V_R(x)$ of a particle with mass m moving along the trajectory $x = x(t)$ with the speed $v = \dot{x}$:

$$I = \max_x \int_{t_0}^{t_1} V_R(x) dt_R. \tag{3}$$

where V_R is the relativistic energy and t_R is the time of the observer sitting in the moving particle.

The relativistic approach transforms this problem as follows:

1. Relativistic energy V_R is the potential newtonian potential energy V of the particle plus the added the rest mass energy E.

   $$V_R = V + E$$

   The addition of a constant term E does not change a solution of the variational problem.

2. An outside observer sense a different time that the traveling observer. Time increment dt_R of the traveling particle relates to the time increment dt for an *inertial observer* by Lorentz transform:

   $$dt_R = \sqrt{1 - \frac{v^2}{c^2}} dt \approx \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) dt + O\left(\frac{v^2}{c^2}\right), \quad v = \dot{x}.$$

   where v is the speed of the particle. Here, we use the smallness of the speed v, $v \ll c$, and the Taylor expansion: $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$.

Substituting these quantities into (3), we compute the Lagrangian :

$$I_R = \max_x \int_{t_0}^{t_1} V_R(x) dt_R.$$

Suubstituting these quantities into $V_R dt_R$ in (3), we compute the Lagrangian in time frame of an intertial observer:

$$\begin{aligned} V_R dt_R &= (V + E)\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) dt \\ &= \left(E + V - \underline{\frac{1}{2}E\frac{v^2}{c^2}} + O\left(\frac{v^2}{c^2}\right)\right) dt. \end{aligned} \tag{4}$$

If the rest mass energy E is equal to $E = mc^2$, where m is mass and c is the speed of light, the underlined term becomes kinetic energy:

$$\frac{1}{2}Ev^2 = \frac{1}{2}mv^2$$

(recall that $v = \dot{x}$)

In the right-hand side of (5), the rest mass energy E is constant and is not varied, and the last term in small if $v \ll c$. We obtain

$$V_R dt_R - E = \left(V - \underline{\frac{1}{2} m\dot{x}^2} + O\left(\frac{v^2}{c^2}\right)\right) dt. \tag{5}$$

Therefore, the maximization of the relativistic energy leads minimization of the Lagrangian:

$$\frac{1}{2} m\dot{x}^2 - V(x)$$

2 Approximations with penalties

Consider the problem of approximation of a function by another one with better smoothness or other favorable properties. For example, we may want to approximate the noisy experimental curve $f(x)$ by a smooth curve (x). Variational method used for approximations are are follows: A problem is to minimize the the integral norm of the difference $f(x) - u(x)$ plus a penalty imposed on $u(x)$ for being non-smooth. Here we consider several problems of the best approximation.

2.1 Approximation with penalized growth rate

The problem of the best approximation of the given function $h(x)$ by function $u(x)$ with a limited growth rate results in a variational problem

$$\min_u J(u), \quad J(u) = \int_a^b \frac{1}{2}\left(\alpha^2 u'^{\,2} + (h-u)^2\right) dx \tag{6}$$

Here, the first term of the integrand represents the penalty for growth or delay and the second term describes the quality of approximation: the closeness of the original and the approximating functions. The approximation depends on the parameter α: When $\alpha \to 0$, the approximation coincides with $h(x)$ and when $\alpha \to \infty$, the approximation is a constant line, equal to the mean value of $h(x)$.

The equation for function $u(x)$ (Euler equation of (6)) is

$$-\alpha^2 u'' + u = h, \quad u'(a) = u'(b) = 0 \tag{7}$$

Here, the natural boundary conditions are assumed since there is no reason to assign special values of the approximation curve at the ends of the interval. Notice, that Lagrangian satisfies sufficient condition for local minimum: it is convex with respect of u and u'.

If $h(x)$ is periodic, so is $u(x)$. In this case

$$-\alpha^2 u'' + u = h, \quad u(a) = u(b), \;\; u'(a) = u'(b), \quad \text{for perodic } h(x) \tag{8}$$

2.2 Solution of the Euler equation

There are several methods for solution of the linear equation (7) for any $h(x)$

Green's function for approximations with quadratic penalty The Euler equation for the approximation problem is described as a linear problem

$$\mathcal{L}_\alpha(u) = h, \quad \mathcal{L}_\alpha = -\alpha^2 \frac{d^2}{dx^2} + 1.$$

Here, $\mathcal{L}_\alpha$ is the linear operator in the interval $[a, b]$ with the homogeneous boundary conditions (7). $\mathcal{L}_\alpha$ depends on the magnitude α of the penalty, and becomes an identical operator with $\alpha \to 0$.

The problem of an optimal approximation is the problem of inverting this operator:

$$u = \mathcal{L}_\alpha^{-1} h$$

The inversion is called the Green's function (see []) or the resolvent for the operator $\mathcal{L}_\alpha$. The solution of (7) is represented by an integral

$$u(x) = \int_a^b k(x, \xi) h(\xi)\, d\xi$$

where $k(x, y)$ is the kernel that depends on the interval and boundary conditions; it is independent of $h(x)$. We do not derive the Green's function here, referring to books on ODE.

The Green's function is especially simple if $a = -\infty,\ b = \infty$. In this case, the Green's function is []

$$k(x, \xi) = \frac{1}{2\alpha} e^{\frac{|x-\xi|}{\alpha}}$$

Notice, that the best approximation is a weighted average:

$$u(x) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} h(\xi) e^{\frac{|x-\xi|}{\alpha}} d\xi$$

with the exponential kernel $e^{\frac{x-\xi}{\alpha}}$.

One can check that $u(x)$ satisfies the Euler equation, boundary conditions, and continuity and jump conditions at $x = \xi$.

Eigenfunctions approach Another method is based on Fourier series. Assume that $a = 0$ and $x \in [0, b]$. Let us expand $h(x)$ into the cosine series (the half-range expansion, see []):

$$h(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n\, x}{b}\right) \tag{9}$$

where a_n are known coefficients

$$a_n = \frac{2}{b} \int_0^b h(x) \cos\left(\frac{\pi n\, x}{b}\right) dx$$

We are looking for the solution $u(x)$ in the form of series:

$$u(x) = \sum_{n=1}^{\infty} z_n \cos\left(\frac{\pi n\, x}{b}\right)$$

where z_n are some coefficients that need to be defined. The boundary conditions (7) are satisfied. Compute $\mathcal{L}_\alpha u$:

$$\mathcal{L}_\alpha u = -\alpha^2 u'' + u = \sum_{n=1}^{\infty} \left[\left(\frac{\alpha \pi n}{b}\right)^2 + 1\right] z_n \cos\left(\frac{\pi n\, x}{b}\right)$$

Equalizing this series with the series in (9), we find

$$z_n = \frac{a_n}{\left(\frac{\alpha\pi n}{b}\right)^2 + 1}$$

if $\alpha \ll 1$, the intensity of low-frequency components of h and u are close, but the series of u has much smaller magnitude of high-frequency harmonics.

2.3 Approximation with penalized smoothness

The problem of smooth approximation is similar. The penalization functional is designed to penalize function $u(x)$ for being different from a straight line; the penalty is proportional to the integral of the square of the second derivative u''. The best approximation variational problem is

$$\min_u J(u), \quad J(u) = \int_a^b \frac{1}{2}\left(\alpha^4 (u'')^2 + (h-u)^2\right) dx \tag{10}$$

When $\alpha \to 0$, the approximation coincides with $h(x)$ and when $\alpha \to \infty$, the approximation is an affine function $u(x) = a\,x + b$ closest to $h(x)$.

The Euler equation of (10) is

$$\alpha^4 u^{IV} + u = h, \quad u''(a) = u''(b) = 0, \quad u'''(a) = u'''(b) = 0,$$

Here, the natural boundary conditions are assumed since there is no reason to assign special values of the approximation curve at the ends of the interval. If h(x) is periodic, so is $u(x)$.

Similarly to the previous case, we find that the Fourier coefficients of the approximate $u(x)$ are

$$z_n = \frac{a_n}{\left(\frac{\alpha\pi n}{b}\right)^4 + 1}$$

where a_n are Fourier coefficients of $h(x)$.

2.4 Approximation with penalized total variation

Total variation Assume that function $u(x)$ is differentiable, $u(x),\ x \in [a, b]$. The total variation $T(u)$ of u is defined as

$$T(u) = \int_a^b |u'(x)| dx$$

For a monotonic function $u(x)$ the integral is

$$T(u) = \max_{x \in [a,b]} u(x) - \min_{x \in [a,b]} u(x)$$

If $u(x)$ has a finite number N of intervals $L_k = [a_k, b_k]$ of monotonicity then the total variation is

$$T(u) = \sum_k^N \left(\max_{x \in L_k} u(x) - \min_{x \in L_k} u(x) \right) = \sum_k^N |u(a_k) - u(b_k)|$$

Approximation with the total variation penalty This approximation penalizes the approximate u for its total variation. The variational problem with total-variation penalty has the form

$$\min_u J_{TV}(u), \quad J_{TV}(u) = \int_a^b \left[\gamma |u'| + (h - u)^2 \right] dx \tag{11}$$

Here, $\gamma \geq 0$, the first term of the integrand represents the total-variation penalty and the second term describes the closeness of the original curve and its approximation. When $\alpha \to 0$, the approximation coincides with $h(x)$ and when $\alpha \to \infty$, the approximation becomes constant equal to mean value of h.

Because $\frac{\partial \gamma |u'|}{\partial u'} = \gamma \mathrm{sign}\,(u')$, the formal application of Euler equation:

$$(\gamma \mathrm{sign}\,(u'))' + u = h \tag{12}$$

is not helpful because it requires the differentiation of a discontinuous function sign ; besides, the Lagrangian (11) is not a twice-differential function of u' as it is required in the procedure of derivation of the Euler equation.

Analysis of the solution The problem admits a straightforward analysis: Assume for simplicity that $h(x)$ monotonically increases in an interval $[a, b]$, then $u(x)$ is also monotone there. Assume the $u(x)$ varies between some numbers u_1 and u_2,

$$u_1 \leq u(x) \leq u_2$$

Let us call α and β such numbers that $h(\alpha) = u_1$, and $h(\beta) = u_2$.

The cost of the approximation is the sum of four terms. $J = J_1 + J_2 + J_3 + J_4$ The first term is the total variation of u

$$J_1 = \int_a^b \gamma |u'| dx = \gamma (u_2 - u_1)$$

and the next three nonnegative terms are

$$J_2 = \int_a^\alpha (h-u)^2 dx, \quad J_3 = \int_\alpha^\beta (h-u)^2 dx, \quad J_4 = \int_\beta^b (h-u)^2 dx.$$

The second term is zero,

$$J_3 = \int_\alpha^\beta \left((h-u)^2\right) dx = 0$$

if $u(x) = h(x)$ for all x such that $u_1 < u(x) < u_2$. This choice of $u(x)$ does not affect the total variation.

The cost depends on two parameters α and β that have to be chosen to minimize J_{TV}. Function u either coincides with $h(x)$ or stays constant cutting the maximum and the minimum of $h(x)$:

$$u(x) = \begin{cases} h(\alpha) & \text{if } x \in [a, \alpha] \\ h(x) & \text{if } x \in [\alpha, \beta] \\ h(\beta) & \text{if } x \in [\beta, b] \end{cases}$$

Notice that the extremal is broken; the regular variational method based the Euler equation is not effective.

Example 2.1 Consider $h(x) = \sin(x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $\alpha \left(0, \frac{\pi}{2}\right)$ be an a priori unknown number - the cutoff level. By symmetry, we have $\beta = -\alpha$ and $u_1 = -\sin(\alpha)$, $u_2 = \sin(\beta)$. The approximate $u(x)$ is equal to

$$u(x) = \begin{cases} -\sin(\alpha) & \text{if } \; x \in \left[-\frac{\pi}{2}, -\alpha\right] \\ \sin(x) & \text{if } \; x \in [-\alpha, \alpha] \\ \sin(\alpha) & \text{if } \; x \in \left[\alpha, \frac{\pi}{2}\right] \end{cases}$$

The components of the cost of the problems are

$$\begin{aligned} J_1 &= \gamma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |u'| dx = 2\gamma \sin(\alpha), \quad J_3 = 0 \\ J_2 &= J_4 = \int_\alpha^{\frac{\pi}{2}} (\sin(x) - \sin(\alpha)^2 dx, \\ &= \frac{1}{4} \left[(-2\pi + 4\alpha)\cos(\alpha)^2 - 6\sin(\alpha)\cos(\alpha) - 6\alpha + 3\pi\right]. \end{aligned}$$

Notice that J_1 increases and J_2 decreases with α. Optimal value of α is found from the transendental equation

$$\frac{dJ}{d\alpha} = 0 \quad \Rightarrow \; 2\cos(\alpha) + (2\alpha - \pi)\sin(\alpha) = \gamma$$