HW 6

April 12, 2019

1. Write stationary condition (Euler-Lagrange equation and boundary terms) for the problem

$$I = \min_u \frac{1}{2} \int_\Omega (\nabla u)^2 + 2v^T \nabla u) dx + \oint_{\partial\Omega} \left(a\, u(x) + b\, u(s)^2 \right) ds$$

where Ω is a bounded domain in R_2 and $v = v(x)$ is a given vector field and a and b are constants.

2. Derive the Euler-Lagrange equation for the Lagrangian

$$F = F(x, u, \nabla \times u),$$

where $x = (x_1, x_2, x_3)$ is a point in R_3 and $u = (u_1, u_2, u_3)$ is a vector minimizer. Use Stokes theorem for integration by parts. Work on the example

$$F = (\nabla \times u)^T (\nabla \times u) - c^2 u^2,$$

3. By computing Euler-Lagrange equation, show that $\det(\nabla u(x))$ is a Null-Lagrangian, if $x = (x_1, x_2, x_3)$ is a point in R_3 and $u = (u_1, u_2, u_3)$.

4. Write the optimality condition for the minimizer u and the boundary $\partial\Omega$ in the 2D problem

$$I = \min_u \min_\Omega J \quad \text{subject to} \quad \int_\Omega dx = A, \quad u|_{\partial\Omega} = 0$$

where

$$J = \int_\Omega \left(\frac{1}{2} (\nabla u)^2 + q\, u \right) dx$$

and $q(x)$ is a given function.