

Optimal conducting composites

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1 Optimal two-phase design

Consider the optimal design problem for the best structure of a two-component conducting body of minimal or maximal principle conductivity. This problem was used as a testing ground for various methods of structural optimization. To solve it, we introduce several different approaches, which are driven by different arguments but lead to similar results. Each approach has an analog for the one-dimensional variational problem discussed earlier.

The relaxation of an optimal design through composites was suggested in [?], where the Weierstrass conditions were used for the relaxation and the numerical results were obtained in [?]. The problem of an elastic bar of the extremal torsion stiffness was considered. This problem is formally equivalent to the problem of the domain of extremal conductivity.

The relaxation (G -closure) approach was applied to specific design problems in [?, ?] and [?, ?] following earlier research (see, for example, [?, ?]); the convexification of the corresponding nonconvex functional was suggested in [?, ?]; numerical schemes were developed in [?, ?, ?, ?, ?, ?, ?] and other papers. The multicomponent optimal design was considered in [?].

1.0.1 Statement of the Problem

The Principle Conductivity of a Domain Suppose that two materials with different conductivities σ_1 and σ_2 , $\sigma_1 < \sigma_2$, are available. We want to displace these materials in a body that occupies a given domain Ω . The conductivity $\sigma(\mathbf{x})$ at a point \mathbf{x} is equal to

$$\sigma(\mathbf{x}) = \chi(\mathbf{x})\sigma_1 + (1 - \chi(\mathbf{x}))\sigma_2,$$

where $\chi(\mathbf{x})$ is the characteristic function of the domain occupied by the material σ_1 .

The boundary conditions are fixed. Suppose for definiteness that the boundary $\partial\Omega$ consists of three components $\partial\Omega_i$ (see ??) and that the boundary conditions are

$$w \in \mathcal{W}, \quad \mathcal{W} = \{w : w|_{\partial\Omega_1} = 0, w|_{\partial\Omega_2} = 1\}; \quad \mathbf{j} \cdot \mathbf{n}|_{\partial\Omega_3} = 0. \quad (1)$$

Two components $\partial\Omega_1$ and $\partial\Omega_2$ of the boundary are kept at potential values $w = 0$ and $w = 1$, respectively, and $\partial\Omega_3$ is insulated.

The problem of optimal conductivity of a domain. Problem of the best conductivity of a domain f4.10.5

Recall that the energy stored in Ω is equal to the work of the outside forces (potentials) applied on the boundary of the body. In particular, if (1) gives the difference of potentials between two parts of the boundary, then the stored energy is equal to the integral of the normal component of the current $\mathbf{j} \cdot \mathbf{n}$ that passes through the boundary of the domain:

$$I_\chi(\chi) = \int_{\partial\Omega_1} (\mathbf{j} \cdot \mathbf{n}) = \min_{w \in \mathcal{W}} \frac{1}{2} \int_{\Omega} \sigma(\chi)(\nabla w)^2. \quad (2)$$

We call I_χ the *principle conductivity* of the domain Ω . It is naturally defined as the ratio between the principal current and the difference in potentials on the boundary components $\partial\Omega_1$ and $\partial\Omega_2$. Functional I_χ depends on the layout of the materials in Ω described by the characteristic function χ .

We will keep the definition (2) of the principle conductivity in the general setting of the Dirichlet boundary conditions on the components $\partial\Omega_1$ and $\partial\Omega_2$ of the boundary.

Optimal Design Problem Consider the following optimal design problem: Find a layout $\chi(\boldsymbol{x})$ that minimizes the principle conductance I_χ of the domain Ω :

$$\inf_{\chi} I_\chi(\chi), \quad (3)$$

where the cost I is the minimal conductivity of Ω . In the absence of additional constraints, the solution to this problem is trivial; the material with minimal conductivity σ_1 is placed everywhere.

To make the problem nontrivial, we assume that the principle mass M_0 of the first material is fixed:

$$\int_{\Omega} \chi = M_0. \quad (4)$$

Constraint (4) is considered in the standard way by adding (4) with the Lagrange multiplier γ to the doubled energy (3) (we double the energy to avoid repeatedly writing the factor $\frac{1}{2}$ in front of the quadratic form (2)). The problem becomes

$$\min_{\chi} \max_{\gamma} \left[\int_{\Omega} (\sigma(\chi)(\nabla w)^2) + \gamma\chi - \gamma M_0 \right]. \quad (5)$$

Next, we fix the value of the constant γ and solve the problem for the augmented functional J_a that differs from (5) by a constant term $-\gamma M_0$:

$$I = \min_{\chi} \min_{w \in \mathcal{W}} J_a, \quad J_a = \int_{\Omega} [\sigma(\chi)(\nabla w)^2 + \gamma\chi]. \quad (6)$$

The augmented Lagrangian J_a depends on the Lagrange multiplier γ as on a parameter. Different values of γ correspond to different fixed amounts $M_0 \in [0, |\Omega|]$ of the first material. After the solution $w = w(\gamma)$, $\chi = \chi(\gamma)$ of (6) is obtained, we use the constraint (4) to determine γ .

Multipplier γ can be interpreted as the difference between the costs of the two materials. Problem (6) asks for the minimization of the sum of the principle conductivity on the domain and its cost. Notice that $\gamma > 0$ or the solution is trivial: $\sigma_{\text{opt}} = \sigma_1$ everywhere. In other words, we assume that the more expensive material is also less conducting.

1.1 Optimal microstructures

In solving the optimization problem (6), one should take into account possible fine-scale oscillations of the control σ . These oscillations physically mean that

the optimally designed body may tend to become a *composite*. Let us admit that the optimal layout of materials may include composite zones. The composites enlarge the set of admissible controls because they represent limits of rapidly oscillating sequences of the original controls (designs).

We use the homogenization approach for description of fine-scale oscillations of layouts. In other words, we describe a composite by its effective tensor. This approach replaces the set $U = \{\sigma_1, \sigma_2\}$ of admissible materials with the G_m -closure of this set. This way, we take into account all possible fine-scale oscillations of $\chi(\mathbf{x})$.

In dealing with composites, we must determine the best microstructures. The best structure of a composite is obtained from the solution to a so-called *local problem* which is a problem of structural optimization in an infinitesimally small neighborhood of a point of the designed body.

The energy of a highly inhomogeneous medium σ_ε in a small regular domain ω is replaced by the equal energy of an equivalent homogenized medium as follows:

$$\langle \sigma_\varepsilon(\nabla w_\varepsilon)^2 \rangle = \mathbf{e} \cdot \boldsymbol{\sigma}_* \mathbf{e} + o(\|\omega\|), \quad (7)$$

where $\mathbf{e} = \langle \nabla w_\varepsilon \rangle$ and $\langle \cdot \rangle$ is the averaging operator (??).

The total amount of the first material is constrained by (4). The constraint (4) can be replaced by an equivalent integral constraint on the volume fraction m of the first material in the composite, $m = m(\mathbf{x}) \in \mathcal{M}$, where

$$\mathcal{M} = \left\{ m(\mathbf{x}) \in [0, 1], \quad \int_{\Omega} m(\mathbf{x}) = M \right\}. \quad (8)$$

The averaged functional in (6) becomes

$$J_a = \int_{\Omega} (\langle \sigma(\chi)(\nabla w)^2 \rangle + \gamma \langle \chi \rangle) = \int_{\Omega} (\mathbf{e} \cdot \boldsymbol{\sigma}_*(m) \mathbf{e} + \gamma m) + o(\|\omega\|).$$

The effective tensor $\boldsymbol{\sigma}_*(m)$ of an optimal composite may vary from point to point together with the field \mathbf{e} , but its value belongs to the G_m -closure: $\boldsymbol{\sigma}_*(m) \in G_m U$.

The optimization problem (6), rounded to $\|\omega\|$, becomes:

$$I = \min_{m \in \mathcal{M}} \min_{\mathbf{e} \in \mathcal{E}} \min_{\boldsymbol{\sigma}_* \in G_m U} \int_{\Omega} (\mathbf{e} \cdot \boldsymbol{\sigma}_* \mathbf{e} + \gamma m) \quad (9)$$

where

$$\mathcal{E} = \{ \mathbf{e} : \mathbf{e} = \nabla w, \quad w \in \mathcal{W} \}.$$

It is called the *relaxed problem*. Note that the relaxed problem does not have rapidly oscillating minimizing sequences of layouts because the G -limits of these sequences are already included in the set of admissible controls.

The internal operation $\min_{\boldsymbol{\sigma}_* \in G_m U}$ asks for the best structure of a composite with a fixed fraction m submerged into a fixed field \mathbf{e} . The next operation $\min_{\mathbf{e}_0 \in E}$ defines the field in the domain Ω if the structure is chosen optimally,

but the layout of the volume fraction $m(\mathbf{x})$ is somehow assigned. The last operation $\min_{m \in \mathcal{M}}$ determines the layout $m(\mathbf{x})$ subject to the integral constraint (8). The order of the minimal operations can be chosen arbitrarily.

The variational method allows us to derive the bounds for coefficients of the effective tensor. Indeed, any admissible trial function $\mathbf{e}_{\text{trial}}(\mathbf{x})$ that satisfies (??) provides an upper bound for a diagonal coefficient of $\boldsymbol{\sigma}_*$ due to (??).

The simplest bound is given by a *constant* trial function

$$\mathbf{e}_{\text{trial}}(\mathbf{x}) = \text{constant}(\mathbf{x}) = \mathbf{i}_1 \quad \forall \mathbf{x} \quad (10)$$

that obviously belongs to the set \mathcal{E} (see (??)). If we substitute $\mathbf{e}_{\text{trial}}$ into (??) and recall that $\boldsymbol{\sigma}(\mathbf{x}) = \sigma(\mathbf{x})I$, we obtain

$$(\boldsymbol{\sigma}_*)_{11} \leq \langle \mathbf{i}_1 \cdot \boldsymbol{\sigma} \mathbf{i}_1 \rangle = \langle \boldsymbol{\sigma}_{11} \rangle. \quad (11)$$

Varying the orientation of the vector of \mathbf{i} , we obtain the matrix inequality:

$$\boldsymbol{\sigma}_* \leq \langle \boldsymbol{\sigma} \rangle. \quad (12)$$

Particularly, the maximal eigenvalue of $\boldsymbol{\sigma}_*$ is bounded from above by the maximal eigenvalue of $\langle \boldsymbol{\sigma} \rangle$.

For a composite assembled from several materials with volume fractions m_i and conductivity tensors $\boldsymbol{\sigma}_i$ we have

$$\langle \boldsymbol{\sigma} \rangle = \sum_{i=1}^N m_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_a, \quad (13)$$

where subindex a denotes the arithmetic mean. The bound (12) is called the *Reuss bound* [?] or the *arithmetic mean bound*.

The dual variational principle (Thompson's principle) also determines a bound for the effective tensor $\boldsymbol{\sigma}_*$. The diagonal coefficient β_*^{11} of the inverse tensor $\boldsymbol{\beta} = \boldsymbol{\sigma}^{-1}$ is

$$\beta_*^{11} = \min_{\mathbf{j} \in \mathcal{J}} \langle \mathbf{j} \cdot \boldsymbol{\sigma}^{-1} \mathbf{j} \rangle, \quad (14)$$

where

$$\mathcal{J} = \{ \mathbf{j} : \nabla \cdot \mathbf{j} = 0, \quad \langle \mathbf{j} \rangle = \mathbf{i}_1, \quad \mathbf{j} \text{ is 1-periodic} \}. \quad (15)$$

Thompson's principle leads to upper estimates of the coefficients of the inverse tensor $\boldsymbol{\sigma}_*^{-1}$ (which are the *lower* estimates of the tensor $\boldsymbol{\sigma}_*$). Again, using the constant trial function, one obtains the inequality

$$\beta_*^{11} \leq \langle \mathbf{i}_1 \cdot \boldsymbol{\sigma}^{-1} \mathbf{i}_1 \rangle,$$

which leads to

$$\boldsymbol{\sigma}_*^{-1} \leq \langle \boldsymbol{\sigma}^{-1} \rangle = \sum_{i=1}^N m_i \boldsymbol{\sigma}_i^{-1} = \boldsymbol{\sigma}_h^{-1}, \quad (16)$$

where

$$\boldsymbol{\sigma}_h = \left(\sum_{i=1}^N m_i \boldsymbol{\sigma}_i^{-1} \right)^{-1} \quad (17)$$

denotes the harmonic mean. This bound is called the *Voigt bound* [?] or the *harmonic mean bound*.

Together, inequalities (12) and (17) provide two-sided bounds of the range of variation of the effective properties tensor:

$$\boldsymbol{\sigma}_h \leq \boldsymbol{\sigma}_* \leq \boldsymbol{\sigma}_a. \quad (18)$$

The range $[\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_a]$ is called the *Wiener box*. It depends only on the properties of the original materials and their fractions in the composite. The inequalities (18) are valid for any composite regardless of its geometry; we call them geometrically independent bounds. These inequalities are also called *Wiener inequalities* [?].

Remark 1.1 Similar bounds can be established for other equilibria that satisfy a minimum variational principle. Indeed, the constant trial function similar to (10) trivially satisfies any linear differential restrictions.

Note that the Wiener bounds are invariant to interchanging the properties tensors with their inverses:

$$(\boldsymbol{\sigma}^{-1})_h \leq \boldsymbol{\sigma}_*^{-1} \leq (\boldsymbol{\sigma}^{-1})_a. \quad (19)$$

The equivalence follows from obvious identities

$$(\boldsymbol{\sigma}^{-1})_h = (\boldsymbol{\sigma}_a)^{-1}, \quad (\boldsymbol{\sigma}^{-1})_a = (\boldsymbol{\sigma}_h)^{-1}. \quad (20)$$

They demonstrate that the upper bound for the “direct” tensor $\boldsymbol{\sigma}$ becomes the lower estimate for the inverse tensor $\boldsymbol{\sigma}^{-1}$ and vice versa.

The Local Problem: Lower Bound We start with the inner minimization problem in the infinitesimal neighborhood $\omega(\mathbf{x})$ of a point \mathbf{x} :

$$\min_{\boldsymbol{\sigma}_* \in G_m U} \mathbf{e} \cdot \boldsymbol{\sigma}_* \mathbf{e}. \quad (21)$$

We have not described the G_m -closure set for the conductivity problem.¹ Fortunately, problem (21) can be solved without the complete description of that set [?, ?].

First, notice that the orientation of the effective tensor $\boldsymbol{\sigma}_*$ is arbitrary because an optimal structure can be arbitrarily rotated. The optimal orientation of $\boldsymbol{\sigma}_*$ is realized when the eigenvector that corresponds minimal eigenvalue λ_{\min} is co-directed with \mathbf{e} . The quadratic form $\mathbf{e} \cdot \boldsymbol{\sigma}_* \mathbf{e}$ becomes

$$\min_{\text{orientation}} \mathbf{e} \cdot \boldsymbol{\sigma}_* \mathbf{e} = \lambda_{\min} \mathbf{e}^2. \quad (22)$$

¹Actually, this set is described in Chapter 11.

Next, the optimal structure must possess the minimal value of λ_{\min} among all microstructures. Recall (see (??)) that all eigenvalues of σ_* vary in the interval $[\sigma_h, \sigma_a]$. Particularly, the minimal eigenvalue does not exceed the harmonic mean σ_h of mixed conductivities:

$$\lambda_{\min} \geq \sigma_h, \quad \sigma_h = \frac{\sigma_1 \sigma_2}{m \sigma_2 + (1 - m) \sigma_1}. \quad (23)$$

Therefore, the minimum in (21) in any infinitely small region ω is bounded from below:

$$\min_{\sigma_* \in G_m U} \mathbf{e} \cdot \sigma_* \mathbf{e} \geq \sigma_h \mathbf{e}^2. \quad (24)$$

The last inequality demonstrates the *sufficient optimality conditions* for the stored energy.

Attainability of the Bound The bound (24) is attainable: It corresponds to a laminate structure where laminates are oriented along the field. Indeed, the harmonic mean of the conductivities is exactly the effective conductivity of laminates in that direction. Therefore, optimal structures can be imitated by properly oriented laminates.

The Relaxed Problem in Large We obtain the formulation of the relaxed problem by substitution of the relaxed Lagrangian (24) into the minimization problem (9):

$$I = \min_{m \in \mathcal{M}} \min_{\mathbf{e} = \nabla w} \int_{\Omega} (\sigma_h(m) \mathbf{e}^2 + \gamma m), \quad \sigma_h(m) = \frac{\sigma_1 \sigma_2}{m \sigma_2 + (1 - m) \sigma_1}. \quad (25)$$

The isotropy of the relaxed problem is expected, because the optimal structure is chosen among all structures of arbitrary orientation, and the bound (24), therefore, is independent of direction.

Remark 1.2 The minimal energy stored in anisotropic laminate structures is equal to the energy of an isotropic material with conductivity σ_h . This equivalence was used in a numerical scheme [?] to simplify the calculations. Namely, we replaced the optimally oriented anisotropic composite by the isotropic material with conductivity $\sigma_h(m)$ and numerically found the best layout of m . After the numerical solution was found, we easily determined the laminate composite with the same energy and used the same amount of materials as the isotropic medium σ_h .

The relaxation is successful due to

1. the available geometrically independent bound (the harmonic mean bound), and
2. the known optimal structure (laminates) that realizes the bound.

Solution to the Relaxed Problem

Lagrangian The relaxed problem can be solved by a standard technique of the calculus of variations. First, we establish necessary conditions of optimality. We change the sequence of minimal operations and minimize the integrand $\Phi(\mathbf{e})$ of (25) over m with the “frozen” field \mathbf{e} :

$$\Phi(\mathbf{e}) = \min_{m \in [0,1]} \left(\frac{\sigma_1 \sigma_2}{m \sigma_2 + (1-m) \sigma_1} \mathbf{e}^2 + \gamma m \right). \quad (26)$$

The optimal value m_0 of m is expressed through the field \mathbf{e} and is equal to

$$m_0 = \begin{cases} 0 & \text{if } |\mathbf{e}| \leq \frac{C}{\sigma_2}, \\ -\frac{\sigma_1}{\sigma_2 - \sigma_1} + \frac{C}{\gamma} |\mathbf{e}| & \text{if } |\mathbf{e}| \in \left[\frac{C}{\sigma_2}, \frac{C}{\sigma_1} \right], \\ 1 & \text{if } |\mathbf{e}| \geq \frac{C}{\sigma_1}, \end{cases} \quad (27)$$

where

$$C = \sqrt{\frac{\gamma \sigma_1 \sigma_2}{\sigma_2 - \sigma_1}}.$$

This condition says that the volume fraction of σ_1 decreases when the density of the field increases until it reaches the boundaries of its range. The Lagrangian $\Phi(\mathbf{e})$ is

$$\Phi(\mathbf{e}) = \begin{cases} \sigma_1 \mathbf{e}^2 + \gamma & \text{if } |\mathbf{e}| \leq \frac{C}{\sigma_2}, \\ -\gamma \frac{\sigma_1}{\sigma_2 - \sigma_1} + 2C |\mathbf{e}| & \text{if } |\mathbf{e}| \in \left[\frac{C}{\sigma_2}, \frac{C}{\sigma_1} \right], \\ \sigma_2 \mathbf{e}^2 & \text{if } |\mathbf{e}| \geq \frac{C}{\sigma_1}. \end{cases} \quad (28)$$

Remark 1.3 The energy $\Phi(\mathbf{e})$ in the composite zone is an affine function of \mathbf{e} . This property deserves a physical explanation because the energy of a linear conducting composite is a quadratic function of \mathbf{e} . To explain the linearity of the energy of an optimal composite we observe that the increase of the magnitude of the field \mathbf{e} leads to a change in the structure of the optimal composite (here, to a decrease in the volume fraction m). The variation of the conductivity of the optimal composite partly compensates the increase of the energy with the magnitude of the field.

We also find from (28) that the magnitude $|\mathbf{j}|$ of the current

$$\mathbf{j} = \frac{\partial \Phi}{\partial \mathbf{e}} = \sigma_h \mathbf{e}$$

is constant in the composite zone: $|\mathbf{j}| = 2C$. This condition expresses a qualitative property of an optimal design: it evenly distributes acting fields throughout the domain.

Optimal Solution The optimal solution w ($\nabla w = \mathbf{e}$) is the solution of the Euler equation to the variational problem

$$I = \min_w \int_{\Omega} \Phi(\nabla w), \quad (29)$$

where w also satisfies the boundary conditions. If m reaches its bounds ($m = 0$ or $m = 1$), then the composite becomes pure materials and w satisfies the Laplace equation

$$\Delta w = 0 \begin{cases} \text{if } |\nabla w| \leq \frac{C}{\sigma_2} \gamma, \\ \text{if } |\nabla w| \geq \frac{C}{\sigma_1} \gamma. \end{cases} \quad (30)$$

In the composite zone, $m \in (0, 1)$, the Lagrangian $\Phi(\mathbf{e})$ is an affine function of $|\mathbf{e}|$, (28). The second-order Euler–Lagrange equation degenerates into the system of two nonlinear first-order equations.

1. The current has the constant magnitude and is divergencefree

$$|\mathbf{j}| = 2C, \quad \nabla \cdot \mathbf{j} = 0.$$

The first equation implies that \mathbf{j} can be represented through a scalar function ϕ as

$$\mathbf{j}(\phi) = 2C(\cos \phi, \sin \phi) \quad (31)$$

and the second equation states that ϕ satisfies the first-order partial differential equation

$$-\sin \phi \frac{\partial \phi}{\partial x_1} + \cos \phi \frac{\partial \phi}{\partial x_2} = 0,$$

which follows from $\nabla \cdot \mathbf{j} = 0$.

2. The constitutive relation $\mathbf{e} = \frac{1}{\sigma_n} \mathbf{j}$ expresses the curlfree field $\mathbf{e} = \nabla w$ ($\nabla \times \mathbf{e} = 0$) through the control σ_h .

$$\nabla \times \left(\frac{\mathbf{j}(\phi)}{\sigma_h} \right) = 0.$$

This relation serves to find the control.

Observe that the second-order elliptical equation of conductivity splits into two nonlinear first-order equations for ϕ and m . In other words, the equation for the optimal conductor reaches the boundary of the ellipticity.

In solving these equations we obtain a solution $w(\gamma), \sigma_h(\gamma)$ that depends only on γ . Finally, we choose γ to satisfy the integral constraint (4) on the available amounts of materials.

Now we can solve the problem of an optimal distribution of two isotropic material in a domain stated at the beginning of this chapter. We use the results of previous chapters, Namely,

1. We recognize a possibility of composite materials with variable properties in an optimal layout.
2. Generally, optimal composite is anisotropic; it optimally oriented in each point according to results in Section ??.
3. The low bound for the minimal eigenvalue and the upper bound for the maximal eigenvalue is the Wiener bounds, according to Section ??.

These points allow for the formulation of optimality conditions for a structural optimization problem.

Optimal Structures An appropriately oriented laminate provides the minimal value of J . Indeed, the laminates have simultaneously the maximal conductivity σ_a in a direction(s) (along the layers) and the minimal conductivity σ_h in the perpendicular direction (across the layers).

The optimal laminates are oriented so that the normal \mathbf{n} coincides with the direction of the field \mathbf{a} , and the tangent \mathbf{t} coincides with the direction of \mathbf{b} . The cost $J(\boldsymbol{\sigma}_{\text{lam}})$ of the local problem for laminate structure $\boldsymbol{\sigma}_{\text{lam}}$ coincides with the bound (??).

1.2 Solution in the large

The solution to the auxiliary local problem allows us to compute K from (??). We denote $m_1 = m$, $m_2 = 1 - m$; assume that $\sigma_2 > \sigma_1$; and calculate an optimal value of the volume fractions of materials in the laminates. We have

$$\begin{aligned} \frac{K}{|\mathbf{p}||\mathbf{q}|} &= \min_{m \in [0,1]} (\lambda_{\min}(m)\mathbf{a}^2 - \lambda_{\max}(m)\mathbf{b}^2) \\ &= \min_{m \in [0,1]} \left(\frac{\sigma_1\sigma_2}{m\sigma_2 + (1-m)\sigma_1} \mathbf{a}^2 + (m\sigma_1 + (1-m)\sigma_2)\mathbf{b}^2 \right). \end{aligned} \quad (32)$$

The optimal value m^{opt} of m depends only on the ratio between $|\mathbf{a}|$ and $|\mathbf{b}|$,

$$\frac{|\mathbf{a}|}{|\mathbf{b}|} = \cot \theta, \quad (33)$$

and is equal to

$$m^{\text{opt}} = \begin{cases} 0 & \text{if } \cot \theta \leq \sqrt{\frac{\sigma_1}{\sigma_2}}, \\ \frac{\sqrt{\sigma_1\sigma_2}}{\sigma_2 - \sigma_1} \left(\cot \theta - \sqrt{\frac{\sigma_1}{\sigma_2}} \right) & \text{if } \sqrt{\frac{\sigma_1}{\sigma_2}} \leq \cot \theta \leq \sqrt{\frac{\sigma_2}{\sigma_1}}, \\ 1 & \text{if } \cot \theta \geq \sqrt{\frac{\sigma_2}{\sigma_1}}. \end{cases} \quad (34)$$

Equation (34) says that the optimal concentration of materials in the laminates depends only on the angle θ .

We find the optimal value of the functional $K = J(m^{\text{opt}})$:

$$\frac{K}{|\mathbf{p}||\mathbf{q}|} = \begin{cases} \sigma_2 \cos 2\theta & \text{if } \cot \theta \leq \sqrt{\frac{\sigma_1}{\sigma_2}}, \\ (\sigma_1 + \sigma_2) \sin^2 \theta & \text{if } \sqrt{\frac{\sigma_1}{\sigma_2}} \leq \cot \theta \leq \sqrt{\frac{\sigma_2}{\sigma_1}}, \\ \sigma_1 \cos 2\theta & \text{if } \cot \theta \geq \sqrt{\frac{\sigma_2}{\sigma_1}}. \end{cases} \quad (35)$$

To complete the solution, it remains to pass to the original notation $\nabla \langle w \rangle = \mathbf{p}$ and $\nabla \langle \lambda \rangle = \mathbf{q}$, substitute the value of the local problem into the functional (??), and find the Euler–Lagrange equations of the problem:

$$I_A = \min_{\langle w \rangle} \max_{\langle \lambda \rangle} \int_{\mathcal{O}} [F(\langle w \rangle) + \langle \lambda \rangle q + K]. \quad (36)$$

Note that the equations for $\langle w \rangle$ and $\langle \lambda \rangle$ are coupled because the optimal properties depend on both of them.

Numerical Procedure Practically, we have used a different procedure for the numerical solution; see [?]. The iterative method has been organized as follows:

1. Given a layout of σ , we compute the solution w of problem (??) and the solution λ of the adjoint problem (??).
2. The optimal layouts $m(\mathbf{x})$ and $\theta(\mathbf{x})$ is found from (33). Then we return to the first step.