I. Basics of the theory

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1 Origin

We know the exact date when Calculus of Variations was born. It started with the challenge posed by Johann Bernoulli in Acta Eruditorum in June 1696. He introduced the problem as follows:

> *I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solutions will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, and others, I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*

The posed problem was called brachistochrone (from Greek brakhistos, superlative of brakhus short + chronos time) and was formulated in the following way:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Within a year five mathematicians responded: Jacob Bernoulli (brother of Johann and the primary target of the challenge), Gottfried Wilhelm Leibniz, Isaac Newton, and Guillaume de L'Hôpital (indeed, they were the most brilliant mathematicians in the world!). Together with Johann Bernoulli himself, they created methods for funding this "best curve." The sought curve out to be a cycloid, the solution discussed below, in Section **??**. The methods were slightly different, but all of them used specifics of the suggested problem. For further details of this remarkable story, see, for example, the paper.[1]

Seventy years later, in 1766, Leonhard Euler developed the regular method to find optimal curves. More precisely, he found a necessary condition for optimality of curves, analogous to the condition of vanishing of derivative of a function in an extremal point. He proved that optimal curve satisfies a differential equation called now Euler or Euler-Lagrange equation (the contribution of Lagrange we discuss later).

> *Since, however, the rules ⟨for isoperimetric curves or, in modern terms, extremal problems⟩ were not sufficiently general, the famous Euler undertook the task of reducing all such investigations to a general method which he gave in the original work in which the profound science of the calculus shines through. Even so, while the method is ingenious and rich, one must admit that it is not as simple as one might hope in a work of pure analysis.*

[1] The Brachistochrone Problem: Mathematics for a Broad Audience via a Large Context Problem, by Jeff Babb and James Currie, TMME, vol5, nos.2&3, p.169

In "Essay on a new method of determining the maxima and minima of indefinite integral formulas", by Lagrange, 1760

2 Stationarity: First variation

The technique was developed by Euler, who also introduced the name "Calculus of variations" in 1766. The method is based on an analysis of infinitesimal variations of a minimizing curve.

2.1 Euler equation

The main scheme of the variational method is as follows: Assume that the optimal curve $u(x)$ exist among smooth (twice-differentiable curves), $u \in C_2[a, b]$. Compare the optimal curve with close-by trajectories $u(x) + \delta u(x)$, where $\delta u(x)$ is small in some sense. Using the smallness of δu, simplify the comparison, deriving necessary conditions for the optimal trajectory $u(x)$. Variational methods give only necessary conditions of optimality because it is assumed that the compared trajectories are close to each other; nevertheless, they apply to a great variety of extremal problems.

Consider the problem called *canonic* or the *simplest* problem of the calculus of variations

$$\min_u I(u), \quad I(u) = \int_0^1 F(x, u, u')dx, \quad u(0) = a_0, \; u(1) = a_1. \tag{1}$$

where F is called Lagrangian. It is assumed to be is twice differentiable function of its three arguments. The method is as follow:

We assume that function $u_0 = u_0(x)$ is a minimizer and replace u_0 with a test function $u_0 + \delta u$, The test function $u_0 + \delta u$ satisfies the same boundary conditions as u_0. If indeed u_0 is a minimizer, the increment of the cost $\delta I(u_0) = I(u_0 + \delta u) - I(u_0)$ is nonnegative:

$$\delta I(u_0) = \int_0^1 (F(x, u_0 + \delta u, (u_0 + \delta u)') - F(x, u_0, u_0'))dx \geq 0. \tag{2}$$

If δu is not specified, the equation (2) is usually not informative, because $u_0(x) + \delta u(x)$ is an arbitrary curve. If, however, we assume that the norm$\|\delta u\|$ of the variation δu is infinitesimal, we define a local minimizer. Calculus of variations suggests a set of tests that differ by types of variations δu and corresponding form of (2).

Euler–Lagrange Equations The simplest condition of optimality (the Euler–Lagrange equation) is derived assuming that the variation δu is infinitesimally small and localized:

$$\delta u = \begin{cases} \rho(x) & \text{if} \quad x \in [x_0, x_0 + \varepsilon], \\ 0 & \text{if} \quad x \text{ is outside of } [x_0, x_0 + \varepsilon]. \\ \rho(x_0 \pm \varepsilon) & = 0 \end{cases} \tag{3}$$

Here $\rho(x)$ is a continuous function that vanishes at points x_0 and $x_0 + \varepsilon$ and is constrained as follows:

$$|\rho(x)| < \varepsilon, \quad |\rho'(x)| < \varepsilon \quad \forall x \in (0,\, 1). \tag{4}$$

The integrant F at the perturbed trajectory can be expended into Taylor series,

$$F(x, u_0 + \delta u, (u_0 + \delta u)') = F(x, u_0, \delta u_0') + \left.\frac{\partial F}{\partial u}\right|_{x, u_0, \delta u_0'} \delta u \;\; + \left.\frac{\partial F}{\partial u'}\right|_{x, u_0, \delta u_0'} \delta u' + o(\delta u, \delta u')$$

where $o(\delta u, \delta u')$ denotes terms that are much smaller than δu and $\delta u'$ when $\varepsilon \to 0$. Substituting this expression into (2) we obtain

$$\delta I(u_0) = \int_0^1 \left(\frac{\partial F}{\partial u}(\delta u) + \underline{\frac{\partial F}{\partial u'}(\delta u)'} \right) dx + o(\varepsilon) \geq 0. \tag{5}$$

The variations δu and $(\delta u)'$ are related, and $(\delta u)'$ can be excluded. Integration by parts of the underlined term in (5) gives

$$\int_0^1 \frac{\partial F}{\partial u'}(\delta u)' dx = \int_0^1 \left(-\frac{d}{dx}\frac{\partial F}{\partial u'} \right) \delta u \, dx + \left.\frac{\partial F}{\partial u'}\delta u \right|_{x=0}^{x=1} \tag{6}$$

and we obtain

$$0 \leq \delta I(u_0) = \int_0^1 S_F(u_0) \delta u \, dx + \left.\frac{\partial F}{\partial u'}\right|_{x=1} \delta u(1) - \left.\frac{\partial F}{\partial u'}\right|_{x=0} \delta u(0) + o(\varepsilon), \tag{7}$$

where we S denotes the expression

$$S_F(u) = -\frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{\partial F}{\partial u}. \tag{8}$$

The nonintegral term in the right-hand side of (7) is zero, because the boundary values of u are prescribed $u(0) = a_0$ and $u(1) = a_1$; therefore their variations $\delta u|_{x=0}$ and $\delta u|_{x=1}$ equal zero,

$$\delta u(0) = 0, \quad \delta u(1) = 0.$$

Due to the arbitrariness of the sign of δu in the integral in the right-hand side of (7), we conclude that $S_F(u_0) = 0$:

Theorem 2.1 (Stationarity) Any differentiable and bounded minimizer u_0 of the variational problem (1) is a solution to the boundary value problem

$$S_F(u) = \frac{d}{dx}\frac{\partial F}{\partial u'} - \frac{\partial F}{\partial u} = 0 \quad \forall x \in (0,1); \quad u(0) = u_0, \quad u(1) = u_1, \tag{9}$$

called the *Euler–Lagrange equation*.

The Euler–Lagrange equation is also called the *stationary condition* since it expresses stationarity of the minimizing curve$u_0(x)$. In the next, we will omit symbol $_0$ for the optimal curve if this is not ambiguous.

Remark 2.1 The stationarity test alone does guarantee that $u(x)$ is a true minimizer or that a solution to (9) exists. It says that if the minimizer is differentiable, (9) holds. In the same time, the function u that *maximizes* $I(u)$ satisfies the same Euler–Lagrange equation if it is differentiable. The tests that distinguish minimal trajectory from other stationary trajectories are discussed in Section **??**.

In this derivation, we assume that the extremal $u(t)$ is a twice differentiable function of x; it can be rewritten in another form. Applying the chain rule

$$\frac{d}{dx}Z(x,u,u') = \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial u}u' + \frac{\partial Z}{\partial u'}u''$$

to $Z = \frac{\partial F}{\partial u'}$ we write the left-hand side of equation (9) as

$$S_F(u) = \frac{\partial^2 F}{\partial u'^2}u'' + \frac{\partial^2 F}{\partial u' \partial u}u' + \frac{\partial^2 F}{\partial x \partial u'} - \frac{\partial F}{\partial u} \tag{10}$$

Example 2.1 Compute the Euler equation for the problem of the extremal curve that join point $(0,0)$ and $(1,a)$ and minimizes the sum of its quadratic deviation and the quadratic deviation of its derivative:

$$I = \min_{u(x)} \int_0^1 \left(\frac{1}{2}(u')^2 + \frac{1}{2}u^2\right) dx \quad u(0) = 0,\ u(1) = a$$

We compute $\frac{\partial F}{\partial u'} = u'$, $\quad \frac{\partial F}{\partial u} = u$ and the Euler equation becomes

$$u'' - u = 0 \ \text{ in } (0,1), \quad u(0) = 1,\ u(1) = a.$$

The solution is $u_0(x) = Ae^x + Be^{-x}$, the constants A and B are found from boundary conditions, $A = -B,\ A\,e + B\,e^{-1} = a$. The solution is

$$u(x) = \frac{a\sinh(x)}{\sinh(1)}.$$

2.2 First integrals: Three special cases

In several cases, the Euler equation (9) can be integrated at least once. These are the cases when Lagrangian $F(x,u,u')$ does not depend on one of the arguments. Below, we investigate them.

Lagrangian is independent of u' Assume that $F = F(x,u)$, and the minimization problem is

$$J(u) = \int_0^1 F(x,u)dx \tag{11}$$

In this case, the variation does not involve integration by parts and the continuity of the minimizer is not required. Euler equation (9) becomes an algebraic equation for u

$$\frac{\partial F}{\partial u} = 0 \tag{12}$$

Curve $u(x)$ is determined in each point independently of neighboring points. The boundary conditions in (9) are satisfied by a discontinuous function that may jump to the prescribed values $u(0)$ and $u(1)$ in the end points; these conditions do not affect the objective functional.

Example 2.2 Consider the problem

$$\min_{u(x)} J(u), \quad J(u) = \int_0^1 (u - \sin x)^2 dx, \quad u(0) = 1;\ u(1) = 0.$$

The minimal value $J(u_0) = 0$ corresponds to the discontinuous minimizer

$$u(x) = \begin{cases} \sin x & \text{if } \ 0 \le x \le 1 \\ 1 & \text{if } \ x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Remark 2.2 Formally, the discontinuous minimizer contradicts the assumption posed when the Euler equation were derived. To be consistent, we need to repeat the derivation of the necessary condition for the problem (11) without any assumption on the continuity of the minimizer.

Lagrangian is independent of u If Lagrangian is independent on u, $F = F(x, u')$, Euler equation (9) can be integrated once:

$$\frac{\partial F}{\partial u'} = \text{ constant} \tag{13}$$

The first order differential equation (13) for u is the *first integral* of the problem; it defines a quantity that stays constant everywhere along the optimal trajectory. To find the optimal trajectory, it remains to integrate the first order equation (13) and determine the constants of integration from the boundary conditions.

Example 2.3 Consider the problem

$$\min_{u(x)} J(u), \quad J(u) = \int_0^1 (u' - \cos x)^2 dx, \quad u(0) = 1;\ u(1) = 0.$$

The first integral is

$$\frac{\partial F}{\partial u'} = u'(x) - \cos x = C$$

Solving for u, we find the minimizer,

$$u(x) = \sin x + Cx + C_1.$$

The constants C and C_1 are found from and the boundary conditions $u(0) = 1;\ u(1) = 0$ as

$$C_1 = 1, \quad C = -1 - \sin 1,$$

minimizer u_0 and the cost of the problem become, respectively

$$u_0(x) = \sin x - (\sin 1 + 1)x + 1 \quad J(u_0) = (\sin 1 + 1)^2.$$

Notice that the Lagrangian in the example (2.2) is the square of the difference between the minimizer u and the function $\sin x$, and the Lagrangian in the example (2.3) is the square of the difference of their derivatives. In the problem (2.2), the minimizer coincides with $\sin x$ and jumps to the prescribed boundary values. The minimizer u in the example (2.3) does not coincide with $\sin x$ at any interval. The difference between these two examples is that in the last problem the derivative of the minimizer must exist everywhere. The discontinuous minimizer would leave the derivative undefined. An approximation of discontinuous solution would make the derivative to grow in the proximity of the point of discontinuity, this growth would increase the objective functional, and therefore such function is not optimal. We deal with such problems below in Chapter **??**.

Lagrangian is independent of x If $F = F(u, u')$, equation (9) has the first integral:

$$\hat{H}(u, u') = \text{ constant}, \quad \hat{H}(u, u') = u' \frac{\partial F}{\partial u'} - F \tag{14}$$

Remark 2.3 The function $\hat{H}$ plays an important role in the calculus of variation. When it is expressed through u and the impulse $p \frac{\partial F}{\partial u'}$ instead of u, u', it is called Hamiltonian. We deal with properties of hamiltonians later in Chapter .

Indeed, compute the x-derivative of $\hat{H}(u, u')$

$$\frac{d}{dx} \hat{H}(u, u') =$$
$$\left[u'' \frac{\partial F}{\partial u'} + u' \left(\frac{\partial^2 F}{\partial u' \partial u} u' + \frac{\partial^2 F}{\partial u'^2} u'' \right) \right] - \frac{\partial F}{\partial u} u' - \frac{\partial F}{\partial u'} u'' = 0$$

where the expression in square brackets is the derivative of the first term of $\hat{H}(u, u')$. Cancelling the equal terms, we bring this equation to the form

$$u' \left(\frac{\partial^2 F}{\partial u'^2} u'' + \frac{\partial^2 F}{\partial u' \partial u} u' - \frac{\partial F}{\partial u} \right) = 0 \tag{15}$$

which is equal to zero by virtue of Euler equation (10): The expression in parenthesis coincide with the left-hand-side term $S_F(u)$ of the Euler equation in the form (10), simplified for the considered case (F is independent of x, $F = F(u, u')$).

Example 2.4 Consider the Lagrangian

$$F = \frac{1}{2}\left[(u')^2 - \omega^2 u^2\right]$$

The Euler equation is

$$u'' + \omega^2 u = 0$$

The first integral is

$$W = (u')^2 + \omega^2 u^2 = C^2 = \text{ constant}$$

Let us immediately check the constancy of the first integral. The solution u of the Euler equation is

$$u = A\cos(\omega x) + B\sin(\omega x)$$

where A and B are constants. Substituting the solution into the expression for the first integral, we compute

$$\begin{aligned} W = (u')^2 + \omega^2 u^2 \; &= \left[-A\omega\sin(\omega x) + B\omega\cos(\omega x)\right]^2 \\ &+\omega^2\left[A\cos(\omega x) + B\sin(\omega x)\right]^2 = \omega^2(A^2 + B^2) \end{aligned}$$

We have shown that W is constant at the optimal trajectory. In mechanical application, W is the whole energy of the oscillator. Instead of solving the Euler equation, we may solve the first-order equation $W = C$ and obtain the same solution.

2.3 Legendre Test

Stationary conditions point to a possibly optimal trajectory, but they can correspond to minimum, local minimum, maximum, local maximum, or a saddle point of the functional. Now we establish a test that distinguishes local minimum from local maximum or saddle. In addition to being a solution to the Euler equation, the true minimizer satisfies necessary conditions in the form of inequalities.

The conclusion of optimality of the tested stationary curve $u(x)$ is based on a comparison of the problem costs $I(u)$ and $I(u + \delta u)$ computed at u and any close-by admissible curve $u + \delta u$. The closeness of an admissible curve to the optimal one simplifies the calculation and results in conditions that are easy to check.

Consider again the canonical problem of the calculus of variations (1) and assume that the first variation δI is zero that is function $u(x)$ that satisfies the Euler equation and boundary conditions,

$$S_F(u) = 0, \quad u(0) = a, \; u(1) = b, \tag{16}$$

Expanding F into Taylor series and keeping the quadratic terms, we obtain

$$\begin{aligned} \delta I = I(u + \delta u) - I(u) &= \int_a^b (F(x, u + \delta u, u' + \delta u') - F(x, u, u'))dx \\ &= \int_a^b \left(S_F(u)\delta u + A\delta u^2 + 2B\delta u\, \delta u' + C(\delta u')^2\right) dx + \left.\frac{\partial F}{\partial u'}\right|_{x=a}^{x=b}, \end{aligned} \tag{17}$$

where

$$A = \frac{\partial^2 F}{\partial u^2}, \quad B = \frac{\partial^2 F}{\partial u \partial u'}, \quad C = \frac{\partial^2 F}{\partial (u')^2}$$

and all derivatives are computed at the point x_0 at the optimal trajectory $u(x)$. The linear term $S_F(u)\delta u$ in (17) is zero because the Euler equation is satisfied.

The increment $\delta^2 I$ depends on the type of the variation δu used. The Legendre condition corresponds to the following variation:

$$\delta u(x, x_0) = \begin{cases} \epsilon^2 \phi\left(\frac{x-x_0}{\epsilon}\right) & \text{if } \ |x - x_0| < \epsilon \\ 0 & \text{if } \ |x - x_0| \geq \epsilon \end{cases} \tag{18}$$

where $\phi(x)$ is a function with the following properties:

$$\phi(-1) = \phi(1) = 0, \quad \max_{x \in [-1,1]} |\phi(x)| \leq 1, \quad \max_{x \in [-1,1]} |\phi'(x)| \leq 1 \tag{19}$$

The magnitude of this *Legendre-type* variation tends to zero as ϵ^2 when $\epsilon \to 0$, and the magnitude of its derivative

$$\delta u'(x, x_0) = \begin{cases} \epsilon\, \phi'\left(\frac{x-x_0}{\epsilon}\right) & \text{if } \ |x - x_0| < \epsilon \\ 0 & \text{if } \ |x - x_0| \geq \epsilon \end{cases}$$

tends to zero as ϵ^2. The variation is *local*: it is zero outside of the interval of the length 2ϵ. We use these features of the variation in the calculation of the increment of the cost.

Let us estimate the quadratic terms. We have

$$\int_a^b A(x)(\delta u)^2 dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} A(x)(\delta u)^2 dx$$
$$\leq \varepsilon^4 \int_{x_0-\varepsilon}^{x_0+\varepsilon} A(x) dx = 2A(x_0)\,\varepsilon^5 + o(\varepsilon^5)$$

Indeed, the variation δu is zero outside of the interval $[x-\varepsilon, x+\varepsilon]$, has magnitude of the order of ε^2 in this interval, and $A(x)$ is assumed to be continuous at the trajectory. Similarly, we estimate

$$\int_a^b B(x)\delta u\, \delta u'\, dx \leq \varepsilon^3 \int_{x_0-\varepsilon}^{x_0+\varepsilon} B(x) dx = 2B(x_0)\,\varepsilon^4 + o(\varepsilon^4)$$
$$\int_a^b C(x)(\delta u')^2 dx \leq \varepsilon^2 \int_{x_0-\varepsilon}^{x_0+\varepsilon} C(x) dx = 2C(x_0)\,\varepsilon^3 + o(\varepsilon^3)$$

The magnitude of $\delta u'$ is of the order of ε, therefore $|\delta u'| \gg |\delta u|$ as $\varepsilon \to 0$; we conclude that the last term in the integrand in the right-hand side of (17) dominates. The inequality $\delta I > 0$ implies inequality

$$\frac{\partial^2 F}{\partial (u')^2} > 0 \tag{20}$$

which is called *Legendre condition* or *Legendre test*.

Note, that the opposite problem of maximization of integral in (1) corresponds to the same Euler equation and to opposite sign of inequality in the Legendre test.

Example 2.5 Consider the problem in example 2.1. The Lagrangian

$$F = \frac{1}{2}(u')^2 + \frac{1}{2}u^2$$

satisfies the Legendre text, because $\frac{\partial^2 F}{\partial (u')^2} = 1$.

Notice, that the problem of the maximum of the problem with this Lagrangian does not have a differentiable solution. We can show that the cost of this problem is infinite. It corresponds to the limit of sequences of functions, for example, the functions that grow indefinitely at the first half of the interval, and then decrease to meet boundary data.

2.4 Variational problem as a limit of a finite-dimensional optimization problem

Recall the problem of minimization of a differentiable function $f(x)$ of one variable x. The coordinate x_0 where $f(x)$ reaches the minimum satisfies a necessary condition which states that the derivative $f'(x)$ vanishes

$$\text{If } x_0 = \text{argmin} f(x), \text{ then } f'(x_0) = 0$$

Similarly, necessary condition for the minimum of a differentiable function $f(x)$ of n variables $x = [x_1, \dots x_n]$ states that all partial derivatives derivatives $\frac{\partial f(x)}{\partial x_i}$ are zero

$$\text{If } x_0 = \text{argmin} f(x), \text{ then } \left.\frac{\partial f(x)}{\partial x_i}\right|_{x=x0} = 0, \ i = 1, \dots n$$

Let us find similar necessary conditions of optimality for the canonical variational problem (1) considering it as a limit of discrete finite-dimensional problems. Similar approach is used for numerical solution of differential equations.

Discretization Consider a finite-dimensional piece-wise linear approximation of $u(x)$. Divide the interval $[a, b]$ into N subintervals by the points

$$z_0 = a, \ z_1 = a + \Delta, ...z_k = a + k\,\Delta, \ z_N = b, \ k = 1, \dots N, \quad \Delta = \frac{b-a}{N}.$$

Assume that the minimizer u is approximated by a piece-wise linear function $\bar{u}_N(x) \in \mathcal{U}_N$ and denote $u(z_i) = u_i$:

$$\bar{u}(x) \in \mathcal{U}_N, \quad \text{if } \bar{u}(x) = u_i + v(x - z_i) \ \forall x \in [z_i, z_{i+1}]$$

that satisfies boundary conditions, $u_0 = a, n_N = b$. The derivative $v(x) = \hat{u}'(x)$ is a piece-wise constant function

$$v(x) = \mathrm{Diff}\,(\hat{u}_i) \quad \text{if } x \in [z_i, z_{i+1}]$$

where the function Diff is defined at a sequence $[u_1, u_2, ..u_n]$ as:

$$\mathrm{Diff}\,(u_i) = \frac{1}{\Delta}(u_i - u_{i-1});$$

When $\Delta \to 0$, Diff-operator tends to the derivative, $\mathrm{Diff}\,(u_i) \to u'(x_i)$.

The integral in the variational problem (1) can be rewritten as

$$I = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} f(x, u(x), u'(x))dx$$

Functions $u(x)$ and $u'(x)$, in the interval $[z_i, z_{i+1}]$ are approximated by u_i and v_i, respectively, and $f(x, u(x), v(x))$ is approximated by a constant $F_i(u_i, v_i) = f(x_i, u_i, v_i)$. The sum is approximated by the integral sum (Darboux sum)

$$I = I_N = \frac{1}{\Delta} \sum_{i=1}^{N} F_i(u_i, v_i) + O\left(\frac{1}{\Delta}\right)$$

that is completely defined by an N-dimensional vector $\{u_i\}$.

Solution of the discretized problem We find $\min_{u_i} I_N(u_1, \dots, u_N)$. The derivative of I_N with respect to u_i is zero,

$$\frac{d\,I_N}{d\,u_i} = 0. \quad i = 1 \dots, N, \quad \text{If } [u_1, ...u_N]$$

Only two terms, F_i and F_{i+1}, in the above sum depend on u_i:

$$\begin{aligned}
\frac{dF_i}{du_i} &= \frac{\partial F_i}{\partial u_i} + \frac{\partial F_i}{\partial v_i}\frac{1}{\Delta}, \\
\frac{dF_{i+1}}{du_i} &= -\frac{\partial F_{i+1}}{\partial v_{i+1}}\frac{1}{\Delta}. \\
\frac{dF_k}{du_i} &= 0 \quad k \neq i, k \neq i+1
\end{aligned}$$

Therefore,

$$\Delta \frac{\partial I_N}{\partial u_i} = \frac{\partial F_i}{\partial u_u} + \frac{1}{\Delta}\left(\frac{\partial F_i}{\partial v_i} - \frac{\partial F_{i+1}}{\partial v_{i+1}}\right) = 0$$

or, using the definition of Diff-operator,

$$\frac{\partial I_N}{\partial u_i} = \frac{\partial F_i}{\partial u_i} - \mathrm{Diff}\,\frac{\partial F_{i+1}}{\partial v_{i+1}} = 0.$$

Homogenization Finally, we pass to the limit $N \to \infty$, Diff $\to \frac{d}{dx}$, assuming that the sequences

$$\{\bar{u}_i\}_N \text{ and } = \left\{ \frac{\partial F_i}{\partial v_i} \right\}_N$$

of piece-wise linear functions $\{\bar{u}_i(x)\}_N$ and piece-wise constant functions $\left\{ \frac{\partial F_i}{\partial v_i} \right\}_N$ converge to differentiable limits

$$\bar{u}_N(x) \to u(x) \quad \left\{ \frac{\partial F_i}{\partial v_i} \right\}_N \to \frac{\partial f(x,u,u')}{\partial u'} \text{ if } N \to \infty \tag{21}$$

Passing to the limit, we simply replace the index $(.)_i$ with a continuous variable x, vector of values $\{u_k\}$ of the piece-wise constant function with the continuous function $u(x)$, difference operator Diff with the derivative $\frac{d}{dx}$, assuming that all limits exist. Then we obtain

$$\lim_{N\to\infty} \sum_{i=1}^{N} F_i(u_i, \mathrm{Diff}\, u_i) \to \int_a^b F(x,u,u').$$

and

$$\lim_{N\to\infty} \left\{ \frac{\partial F_i}{\partial u_i} - \mathrm{Diff}\left(\frac{\partial F_{i+1}}{\partial \mathrm{Diff}\,(u_{i+1})} \right) \right\}_N \to \frac{\partial F}{\partial u} - \frac{d}{d\,x}\frac{\partial F}{\partial u'}$$

Remark 2.4 The assumptions (21) are not technical. The properties of minimizing sequences are not known *a priori*. Later we discuss conditions on $F(x,u,u')$ that guarantee the differentiability of the limits in (21).

We follow the formal scheme of necessary conditions, thereby tacitly assuming that all derivatives of the Lagrangian exist, the increment of the functional is correctly represented by the first term of its power expansion and the limit of the sequence of finite-dimensional problems exist and does not depend on the partition $\{x_1, \dots x_N\}$ if only $|x_k - x_{k-1}| \to 0$ for all k. We also indirectly assume that the Euler equation has at least one solution consistent with boundary conditions.

If all the made assumptions are correct, we obtain a curve that might be a minimizer because the stationary test cannot disprove it. In other terms, we find that is no other close-by classical curve correspond to a smaller value of the functional. This statement about the optimality seems to be rather weak, but this is what the calculus of variation can give. On the other hand, the variational conditions are universal and, being appropriately used and supplemented by other conditions, lead to a very detailed description of the extremal as we show later in the course.

Remark 2.5 Remark on convergence In the above procedure, we assume that the limits of the components of the vector $\{u_k\}$ represent values of a smooth function in the close-by points $x_1, \dots, x_N$. On the other hand, u_k are solutions of optimization problems with the coefficients that slowly vary with the number k. We need to answer the question of whether the solution of a minimization problem tends to is a differentiable function of x; that is whether the limit

$$\lim_{k\to\infty} \frac{u_k - u_{k-1}}{x_k - x_{k-1}}$$

exists and this is not always the case. We address this question later in Chapter ??

3 Stationarity of boundary terms

3.1 Variation of boundary conditions

Variational conditions and natural conditions So far, we assumed that the boundary conditions in the canonical problem (1) are prescribed. These conditions may be not specified. In this case, they are found by minimization of the functional along wish the minimizer $u(x)$. Also, the objective functional may contain terms defined on the boundary.

Consider the problem where conditions at one of the ends $x = b$ are not fixed and the cost depends on the boundary value u(b).

$$\min_{u(x):u(a)=u_a} I(u), \quad I(u) = \int_a^b F(x, u, u')dx + f(u(b)) \tag{22}$$

The Euler equation for the problem remain the same $S_F(u) = 0$ but this time it must be supplemented by a *variational boundary condition* that is derived from the requirement of the stationarity of the minimizer with respect to variation of the boundary value $\delta u(b)$

$$\delta u \frac{\partial F}{\partial u'} + \delta u \frac{\partial f}{\partial u}$$

The first term comes from the integration by part in the derivation of Euler equation, see (7), and the second is the variation of the boundary term in the objective functional (22) The stationarity condition with respect to the variation of $\delta u(b)$

$$\left.\frac{\partial F}{\partial u'}\right|_{x=b} + \left.\frac{\partial f}{\partial u}\right|_{x=b} = 0 \tag{23}$$

expresses the boundary condition for the extremal $u(x)$ at the endpoint $x = b$. Similar condition can be derived for the point $x = a$ if the value in this point is not prescribed.

Example 3.1 Minimize the functional

$$I(u) = \min_u \int_0^1 \frac{1}{2}(u')^2 dx + Au(1), \quad u(0) = 0$$

Here, we want to minimize the endpoint value and we do not want the trajectory to be steep. The Euler equation $u'' = 0$ must be integrated with boundary conditions $u(0) = 0$ and (see (23)) $u'(1) + A = 0$ The extremal is a straight line, $u = -Ax$. The cost of the problem is $I = -\frac{1}{2}A^2$.

If no out-of-integral terms are presented, the condition becomes

$$\left.\frac{\partial F}{\partial u'}\right|_{x=b} = 0 \tag{24}$$

and it is called the *natural boundary condition.*

Example 3.2 The natural boundary condition for the problem with the Lagrangian $L = (u')^2 + \phi(x, u)$ is $u'|_{x=b} = 0$.

3.2 Broken extremal and the Weierstrass-Erdman condition

The classical derivation of the Euler equation requires the existence of all second partials of F, and the solution u of the second-order differential equation is required to be twice-differentiable.

In some problems, the Lagrangian is only piece-wise twice differentiable; in this case, the extremal consists of several curves – solutions of the Euler equation that are computed at the intervals of smoothness of the Lagrangian. The question is: How to join these pieces together?

We always assume that the extremal u is continuous everywhere and differentiable everywhere except maybe final number of points so that the first derivative u' exists at almost all point of the trajectory. But the derivative u' itself does not need to be continuous to solve the Euler equation: Only the differentiability of $\frac{\partial F}{\partial u'}$ is needed to ensure the existence of the term $\frac{d}{dx}\frac{\partial F}{\partial u'}$ in the Euler equation.

Take again the stationarity condition (**??**) and integrate it from a to x:

$$\int_a^x S_F(u)dz = \int_{x_0}^x \left(\frac{d}{dx}\frac{\partial F}{\partial u'} - \frac{\partial F}{\partial u}\right) dz = 0$$

or

$$\frac{\partial F}{\partial u'} = \int_{x_0}^x \frac{\partial F}{\partial u} dz \tag{25}$$

We obtain the stationarity condition in the integral form.

If $\frac{\partial F}{\partial u}$ is bounded at the optimal trajectory, the right-hand side is a differentiable function of x, and so is the left-hand side. This requirement on differentiability of an optimal trajectory results to the *Weierstrass-Erdman condition on broken extremal.*

At any point of the optimal trajectory, the Weierstrass-Erdman condition must be satisfied:

$$\left[\frac{\partial F}{\partial u'}\right]_-^+ = 0 \quad \text{along the optimal trajectory } u(x) \tag{26}$$

Here $[z]_-^+ = z_+ - z_-$ denotes the jump of a variable z.

Another way to derive the Weierstrass-Erdman conditions is dividing the interval of integration into two subintervals and computing the stationarity of the left and the right part of the trajectory. Doing this, we integrate by parts as in (6) and obtain the boundary terms at the point of breakage

$$\left[\frac{\partial F}{\partial u'}\right]_+ \delta u_+ - \left[\frac{\partial F}{\partial u'}\right]_- \delta u_- = 0$$

The trajectory is continuous at the point of breakage, therefore $\delta u_+ = \delta u_-$ and the condition (26) follows.

Example 3.3 (Broken extremal) Consider the Lagrangian

$$F = \frac{1}{2}a(x)(u')^2 + \frac{1}{2}u^2, \quad a(x) = \begin{cases} a_1 & \text{if } x \in [0, x_*) \\ a_2 & \text{if } x \in [x_*, 1) \end{cases}$$

where x_* is a point in $(0,1)$.

The Euler equation

$$\frac{d}{dx}[a_1 u'] - bu = 0 \quad \text{if } x \in [0, x_*)$$
$$\frac{d}{dx}[a_2 u'] - bu = 0 \quad \text{if } x \in [x_*, 1),$$

holds everywhere in $(0,1)$ except the point x_*,

At $x = x_*$, the Weierstrass-Erdman condition $\frac{\partial F}{\partial u'} = 0$ holds, or

$$a_1 u'(x_* - 0) = a_2 u'(x_* + 0).$$

The derivative u' is discontinuous at x_*; its jump is determined by the jump in the coefficients:

$$u'(x_* + 0) = \left(\frac{a_1}{a_2}\right) u'(x_* - 0)$$

This condition, together with the Euler equation and boundary conditions determines the optimal trajectory.

3.3 Non-fixed interval. Transversality condition

Free boundary Consider now the case when the interval (a, b) is not fixed, and the endpoint must be chosen to minimize the functional. We compute the difference between two functionals

$$\delta I = \int_a^{b+\delta x} F(x, u + \delta u, u' + \delta u')dx - \int_a^b F(x, u, u')dx$$

The linear terms of the difference are

$$\delta I = A_x \delta x + A_u \delta u \tag{27}$$

where A_x is the increment due to variation of the interval when u keeps its stationary value, and A_u is the increment due to variation $\delta_x u = \frac{du}{dx}\delta x$ of u when the interval keeps its stationary value.

Let us compute these quantities. The increment's part A_u is computed in a standard way, considering variation $\delta u(b)$ of the trajectory that is independent of the variation of the interval. We obtain Euler equation $S_F(u) = 0$ supplemented by the boundary term

$$A_u \delta u = \frac{\partial F}{\partial u'}\delta u|_{x=b},$$

This increment is zero either because $u(b)$ is fixed and $\delta u(b) = 0$, or because the natural boundary condition $\frac{\partial F}{\partial u'} = 0$ holds.

The increment A_x consists of two terms. First term $A_x^{(1)}$ is the integral over the added length δx

$$A_x^{(1)} = \int_b^{b+\delta x} F(x, u, u')dx = F(x, u, u')|_{x=b}\delta x$$

the second term $A_x^{(2)}$ is the variation of the condition $\frac{\partial F}{\partial u'}\delta u(b)$ due to shifting the point b to point $b + \delta x$

$$\delta_b u = -u(b + \delta x) + u(b) = -u'|_{x=b}\delta x.$$

Thus the second term is

$$A_x^{(2)} = -\left.\frac{\partial F}{\partial u'}u'\right|_{x=b} \delta x$$

where $\delta_b u$ is the variation of u due to variation of the point b. It is equal

Computing $A_x = A_x^{(1)} + A_x^{(2)}$, we obtain

$$A_x = \left.\left(F(x, u, u') - \frac{\partial F}{\partial u'}u'\right)\right|_{x=b} \delta x.$$

The expression in parentheses is the same as the first integral (**??**); we denote it by $\hat{H}(u, u')$.

Because of the arbitrariness of δx, we end up with two conditions

$$A_x = \left(F(x, u, u') - u'\frac{\partial F}{\partial u'}\right)_{x=b} = 0 \tag{28}$$

and

$$\text{Either } \; \hat{H} = \left.\frac{\partial F}{\partial u'}\right|_{x=b} = 0 \quad \text{or } u = u_b. \tag{29}$$

that are satisfied at the unknown end of the trajectory. Equation (28) together with boundary conditions determine the boundary values of u and the length of the interval of integration, while the equation $S_F(u) = 0$ for $x \in (a, b)$ states that the Euler equation is satisfied along the optimal trajectory.

If the boundary condition is not specified, $\frac{\partial F}{\partial u'} = 0$, and these two equations are simplified to

$$F(x, u, u') = 0, \quad \frac{\partial F}{\partial u'} = 0, \quad \text{at } x = b$$

The second-order differential equation $S_F(u)$ for the extremal has an extra boundary condition (28) to satisfy, but it also has an additional unknown quantity: The non-fixed length of the interval of integration.

Remark 3.1 Notice that the condition at the unknown endpoint $\hat{H} = 0$ has the same form as the first integral $\hat{H} = C$ of the problem in the case when $F(x, u, u')$ is independent of x. Since $\hat{H}$ remains constant along the extremal, the condition (28) cannot be satisfied at an isolated point of the trajectory. Therefore, in this case the problem does not have a solution.

Example 3.4 Consider the problem

$$\min_{u(x),s} \int_0^s \left(\frac{1}{2}u'^2 - u + \frac{x}{2}\right) dx \quad u(0) = 0,$$

Euler equation $u'' + 1 = 0$ and the boundary condition $u(0) = 0$ leads to solution

$$u = -\frac{1}{2}x^2 + Ax, \quad u' = -x + A$$

where A is a constant. The conditions at the unknown endpoint s are

$$\left.\frac{\partial F}{\partial u'}\right|_{x=s} = u' = -s + A = 0 \quad \text{or } A = s$$

(condition (29)):

$$A_x = F(s, u(s), u'(s))|_{u'=0} = -\frac{1}{2}s^2 + \frac{1}{2}s = \frac{1}{2}s(1 - s) = 0$$

(condition (28)). We obtain $s = 1$ and $u = \frac{1}{2}x^2 - x$.

Endpoint at a given curve We may assume that the endpoint lies at a curve $u = \phi(x)$. Then the variations δu and δx are bounded: $\delta u = \phi' \delta x$. The variation (27) of endpoint becomes $A_c = (A_u \phi' - A_x)\delta x$ Since δx is arbitrary, we obtain the boundary condition for the extremal:

$$A_c = F - (u' - \phi')\frac{\partial F}{\partial u'} = 0$$

Example 3.5 Find the shortest distance between the origin and a curve $y = \phi(x)$.

$$\min_{y(x),s} \int_0^s \sqrt{1 + y'^2} dx, \quad y(0) = 0, y(c) = \phi(c)$$

The Lagrangian is $F = \sqrt{1+y'^2}$, it is independent of y.

Euler equation admit the first integral

$$\frac{\partial F}{\partial y'} = C; \quad \frac{y'}{\sqrt{1+y'^2}} = C$$

Therefore, y' is constant, the trajectory is a straight line $y = a\,x$ as expected.

The condition at the meeting point is

$$A_c = \sqrt{1+y'^2} - (y' - \phi')\frac{y'}{\sqrt{1+y'^2}} = 0$$

This expression is simplified to $\phi' y' = -1$. It shows that the optimal trajectory is a line perpendicular to the curve $\phi(x)$ at the meeting point.

3.4 Extremal broken at an unknown point

The problem of en extremal broken in an unknown point is considered similarly. The position of this point is determined from the minimization requirement. Assume that Lagrangian has the form

$$F(x,u,u') = \begin{matrix} F_-(x,u,u') & \text{if } x \in (a,\xi) \\ F_+(x,u,u') & \text{if } x \in (\xi,b) \end{matrix}$$

where ξ is an unknown point in the interval (a,b) of the integration. The Euler equation is

$$S_F(u) = \begin{matrix} S_{F_-}(u) & \text{if } x \in (a,\xi) \\ S_{F_+}(u) & \text{if } x \in (\xi,b) \end{matrix}$$

The stationarity conditions at the unknown point ξ are

$$\frac{\partial F_+}{\partial u'} = \frac{\partial F_0}{\partial u'} \tag{30}$$

(the stationarity of the trajectory) and

$$F_+(u) - u'_+\frac{\partial F_+}{\partial u'} = F_-(u) - u'_-\frac{\partial F_-}{\partial u'} \tag{31}$$

(the stationarity of the position of the transit point). They are derived by the same procedure as the conditions at the end point. The variation δx of the transit point increases the first part of the trajectory and increases the second part, $\delta x = \delta x_+ = -\delta x_-$ which explains the structure of the stationary conditions.

In particular, if Lagrangian is independent of x, the condition (31) express the constancy of the first integral (14) at the point ξ.

Example 3.6 Consider the problem with Lagrangian

$$F(x,u,u') = \begin{matrix} a_+u'^2 + b_+u^2 & \text{if } x \in (a,\xi) \\ a_-u'^2 & \text{if } x \in (\xi,b) \end{matrix}$$

and boundary conditions

$$u(a) = 0, \quad u(b) = 1$$

The Euler equation is

$$S_F(u) = \begin{array}{ll} a_+ u'' - b_- u = 0 & \text{if } x \in (a, \xi) \\ a_- u'' = 0 & \text{if } x \in (\xi, b) \end{array}$$

The solution to this equation that satisfies the boundary conditions is

$$\begin{array}{ll} u_+(x) = C_1 \sinh\left(\sqrt{\frac{b_+}{a_+}}(x-a)\right) & \text{if } x \in (a, \xi) \\ u_-(x) = C_2(x-b) + 1 & \text{if } x \in (\xi, b) \end{array};$$

it depends on three constants ξ, C_1, and C_2 (Notice that the coefficient a_- does not enter the Euler equations). These constants are determined from the thee remaining conditions at the unknown point ξ which express
(1) continuity of the extremal

$$u_+(\xi) = u_-(\xi),$$

(2) the Weierstrass-Erdman condition

$$a_+ u'_+(\xi) = a_- u'_-(\xi),$$

(3) and the transversality condition

$$-a_+(u'_+(\xi))^2 + b_+ u(\xi)^2 = -a_-(u'_-(\xi))^2.$$

Let us analyze them. The transversality condition is the simplest one because it states the equality of two first integral. It is simplified to

$$C_1^2 b_+ = C_2^2 a_-$$

From the condition (2), we have

$$C_1 \sqrt{a_+ b_+} \cosh q = C_2, \quad \text{where } q = \sqrt{\frac{b_+}{a_+}}(\xi - a)$$

Together with the previous condition and the definition of q, it allows for determination of ξ:

$$\cosh q = \sqrt{a_+ a_-}, \quad \Rightarrow \quad \xi = a + \frac{a_+}{b_+} \cosh^{-1} \sqrt{a_+ a_-}$$

Finally, we define constants C_1 and C_2 from the continuity condition:

$$C_1 \sinh q = 1 + C_2(\xi - b)$$

and the transversality condition as

$$C_1 = \frac{\sqrt{a_-}}{\sqrt{a_-}\sinh q - \sqrt{b_+}(\xi - b)}, \quad C_2 = \frac{\sqrt{b_+}}{\sqrt{a_-}\sinh q - \sqrt{b_+}(\xi - b)},$$

4 Geometric optics

4.1 Geometric optics problem

A half of century before the calculus of variations was invented, Fermat suggested that light particles propagate along the trajectory which minimizes the time of travel between the source with coordinates (a, A) and the observer with coordinates (b, B). The Fermat principle implies that light travels along straight lines when the medium is homogeneous and along curved trajectories in an inhomogeneous medium in which the speed $v(x, y)$ of light depends on the position. This principle correctly predicted the refraction of light. (At that time people did not ask how the particles "know" their destination point because of the belief that everything in the world is designed to be the most effective.)

The same problem – minimization of the travel's time – can be formulated as the best route for a cross-country runner; the speed depends on the type of the terrains which the runner crosses; it is a function of the position. This problem is called the problem of geometric optics.

In order to formulate the problem of geometric optics, consider a trajectory in a plane, call the coordinates of the initial and final points of the trajectory (a, A) and (b, B), respectively, assuming that $a < b$ and call the optimal trajectory $y(x)$ thereby assuming that the optimal route is a graph of a function. The time T of travel can be found from the relation $v = \frac{ds}{dt}$ where

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx$$

is the infinitesimal length along the trajectory $y(x)$. We have

$$dt = \frac{ds}{v(x, y)} = \frac{\sqrt{1 + y'^2}}{v(x, y)}dx$$

Then the travel time is expressed through the trajectory and the speed

$$T = \int_a^b dt = \int_a^b \frac{\sqrt{1 + y'^2}}{v(x, y)}dx.$$

Consider minimization of travel time T by choosing the trajectory. The corresponding Lagrangian has the form

$$F(x, y, y') = \psi(y)\sqrt{1 + y'^2}, \quad \psi(x, y) = \frac{1}{v(x, y)};$$

and the Euler equation (1) takes the form

$$\frac{d}{dx}\left(\psi(x, y)\frac{y'}{\sqrt{1 + y'^2}}\right) - \frac{\partial \psi(x, y)}{\partial y}\sqrt{1 + y'^2} = 0, \quad y(a) = A, \; y(b) = B$$

The problem can be further simplified if we assume that the medium is layered and the speed $v(y) = \frac{1}{\psi(y)}$ of travel varies only along the y axis. Then, the Lagrangian is independent of x, the Euler equation admits the first integral $\hat{H}(y, y') = c$ (see (**??**)), or

$$\psi(y)\frac{y'^2}{\sqrt{1+y'^2}} - \psi(y)\sqrt{1+y'^2} = c.$$

The last expression is simplified to:

$$\psi(y) = -c\sqrt{1+y'^2} \tag{32}$$

Solving for y', we obtain the first-order equation with separated variables

$$\frac{dy}{dx} = \pm\frac{\sqrt{\psi^2(y) - c^2}}{c}$$

with the solution

$$x = \pm \int \frac{c\,dy}{\sqrt{\psi^2(y) - c^2}} \tag{33}$$

The equation (32) has a clear geometric interpretation: Derivative y' of the trajectory defines the angle α of the slope of it, $y' = \tan\alpha$. The substitution of this expression into (32) gives

$$\psi(y)\ \cos\alpha = -c \tag{34}$$

which shows that the angle of the optimal trajectory varies with speed $v = \frac{1}{\psi}$ of the signal in the media.

Legendre test Geometric optics problem passes the Legendre test $\frac{\partial^2 F}{\partial y'^2} > 0$

$$\frac{\partial^2}{\partial y'^2}\left(\psi(y)\sqrt{1+y'^2}\right) = \frac{\psi(y)}{(1+y'^2)^{\frac{3}{2}}} > 0$$

that shows that the stationary trajectory corresponds to a local minimum.

Remark 4.1 The solution to the opposite problem of a maximum of travel time should satisfy the same Euler equation, but the Legendre test for this problem is $\frac{\partial^2 F}{\partial y'^2} < 0$ and it is not satisfied. What is wrong?

The solution of this problem does not exist; there are infinitely many wiggly trajectories that correspond to arbitrary long time of travel.

4.2 Snell's law of refraction

Assume that the speed of the signal in a medium is piecewise constant; it changes when $y = y_0$ and the speed v jumps from v_+ to v_-, as it happens on the boundary between air and water,

$$v(y) = \begin{cases} v_+ & \text{if } y > y_0 \\ v_- & \text{if } y < y_0 \end{cases}$$

Let us find what happens with an optimal trajectory. Weierstrass-Erdman condition is written as

$$\left[\frac{\partial F}{\partial u'}\right]_-^+ = 0 \quad \text{or} \quad \left[\frac{y'}{v\sqrt{1+y'^2}}\right]_-^+ = 0$$

Recall that $y' = \tan\alpha$ where α is the angle of inclination of the trajectory to the axis OX, then $\frac{y'}{\sqrt{1+y'^2}} = \sin\alpha$ and we arrive at the expression called Snell's law of refraction

$$\frac{\sin\alpha_+}{v_+} = \frac{\sin\alpha_-}{v_-}$$

4.3 Brachistochrone

The problem of the brachistochrone that Johann Bernoulli put forward in 1696 is formulated as follows: Find the curve of the fastest descent (brachistochrone), the trajectory that allows a mass that slides along it without tension under the force of gravity to reach the destination point in minimal time.

To formulate the problem, we use the law of conservation of the total energy – the sum of the potential and kinetic energy is constant at any time instance:

$$\frac{1}{2}mv^2 + mg(y - y_0) = 0$$

where $y(x)$ is the vertical coordinate of the sought curve. From this relation, we express the speed v as a function of y

$$v = \sqrt{\frac{y - y_0}{2a}}$$

where a is a constant. The problem is reduced to a special case of geometric optics.

Remark 4.2 Johann Bernoulli used Fermat principle to solve brachistochrone; he assumed that the speed is constant in a horizontal layer, and sent the number of layers to infinity

Applying the formula (33), we obtain

$$x = \int \frac{\sqrt{y - y_0}}{\sqrt{2a - (y - y_0)}} dy$$

To compute the quadrature, we change the variable in the integral, setting:

$$y = y_0 + 2a\sin^2\frac{\theta}{2}, \quad dy = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

and find

$$x = 2a\int \sin^2\frac{\theta}{2}d\theta = a(\theta - \sin\theta) + x_0$$

The optimal trajectory is a parametric curve called cycloid

$$\begin{aligned} x &= x_0 + a(\theta - \sin\theta), \\ y &= y_0 + a(1 - \cos\theta), \end{aligned} \tag{35}$$

The cycloid is a curve generated by a motion of a point on ? wheel rim the wheel of radius a rolls on a horizontal line.

Cycloid was a newly discovered curve in the seventeen century. Huygens showed that if a particle slides along the cycloid under the uniform gravity, it takes the same time for the particle to reach the lowest point independently of the starting position of the particle. This property implies that the period of oscillations along the cycloid is independent of the initial position of the particle; this was important for pendulums in clocks.

Cycloid was called isochrone or tautochrone (from Greek prefixes tauto- meaning same or iso- equal, and chrono- time). Johann Bernoulli ended his solution with the remark:

Before I end I must voice once more the admiration I feel for the unexpected identity of Huygens' tautochrone and my brachistochrone. ... Nature always tends to act in the simplest way, and so it here lets one curve serve two different functions, while under any other hypothesis we should need two curves.

In modern terms, brachistochrone is an optimal design problem: the trajectory must be chosen by a designer to minimize the time of travel.

5 Several minimizers

5.1 Euler equations and first integrals

The Euler equation can be naturally generalized to the problem with the vector-valued minimizer

$$I(u) = \min_u \int_a^b F(x, u, u')dx, \tag{36}$$

where $x \in [a, b]$, and $u = (u_1(x), \ldots, u_n(x)]$ is a vector function. We suppose that F is a twice differentiable function of its arguments.

Let us compute the variation $\delta I(u)$ equal to $I(u + \delta u) - I(u)$, assuming that the variation of the extremal and of its derivative is small and localized. To compute the Lagrangian at the perturbed trajectory $u + \delta u$, we use the expansion

$$F(x, u + \delta u, u' + \delta u') = F(x, u, u') + \sum_{i=1}^n \frac{\partial F}{\partial u_i}\delta u_i + \sum_{i=1}^n \frac{\partial F}{\partial u_i'}\delta u_i'$$

We can perform n independent variations of each entry of vector u or variations $\delta_i u = (0, \ldots, \delta u_i \ldots, 0)$. The increment of the objective functional should be zero for each of these variations; otherwise the functional can be decreased by one of them. But the stationary condition for any of the considered variations coincides with the single-minimizer case. We arrive at the system:

$$\delta_i I(u) = \int_a^b \left(\delta u_i \frac{\partial F}{\partial u_i} + \delta u_i' \frac{\partial F}{\partial u_i'} \right) dx \geq 0 \quad i = 1, \dots, n.$$

Proceeding as before, we obtain the system of differential equations of the order $2n$,

$$\frac{d}{dx}\frac{\partial F}{\partial u_i'} - \frac{\partial F}{\partial u_i} = 0, \quad i = 1, \dots n \tag{37}$$

and the boundary terms

$$\sum_{i=1}^{n} \frac{\partial F}{\partial u_i'} \delta u_i \Big|_{x=a}^{x=b} = 0 \tag{38}$$

Remark 5.1 The vector form of the system (37),

$$S_F(u) = \frac{d}{dx}\frac{\partial F}{\partial u'} - \frac{\partial F}{\partial u} = 0, \quad \delta u^T \frac{\partial F}{\partial u'}\Big|_{x=a}^{x=b} = 0 \tag{39}$$

is formally identical to the scalar Euler equation.

Example 5.1 Consider the problem with the Lagrangian

$$F = \frac{1}{2}u_1'^2 + \frac{1}{2}u_2'^2 - u_1 u_2' + \frac{1}{2}u_1^2 \tag{40}$$

The system of stationarity conditions is

$$\begin{aligned}
\frac{d}{dx}\frac{\partial F}{\partial u_1'} - \frac{\partial F}{\partial u_1} &= \quad u_1'' + u_2' - u_1 = 0 \\
\frac{d}{dx}\frac{\partial F}{\partial u_2'} - \frac{\partial F}{\partial u_2} &= \quad (u_2' - u_1)' = 0. \\
u_1 \delta u_1'(z) + (u_2' - u1)\delta u_2(z) &= 0, \ z = a, b
\end{aligned}$$

If consists of two differential equations of second order for two unknowns $u_1(x)$ and $u_2(x)$ and the boundary conditions

First integrals The first integrals that are established for the special cases of the scalar Euler equation, can also be derived for the vector equation.

1. If F is independent of u_k', then one of the Euler equations degenerate into algebraic relation:
$$\frac{\partial F}{\partial u_k} = 0$$
and the order of the system (37) decreases by two. The function $u_k(x)$ can be a discontinuous at an optimal solution. Since the Lagrangian is independent of u_k', the jumps in $u_k(x)$ may occur along the optimal trajectory.

2. If F is independent of u_k, the first integral exists:

$$\frac{\partial F}{\partial u'_k} = \text{constant}$$

   For instance, the second equation in Example 5.1 can be integrated and replaced by

$$u'_2 - u_1 = \text{ constant}$$

3. Finally, if F is independent of x, $F = F(\boldsymbol{u}, \boldsymbol{u}')$ then a first integral exist

$$\hat{H} = u'^T \frac{\partial F}{\partial u'} - F = \text{constant} \tag{41}$$

   Here

$$u'^T \frac{\partial F}{\partial u'} = \sum_{i=1}^{n} u'_i \cdot \frac{\partial F}{\partial u'_i}$$

   For the Lagrangian in Example 5.1, this first integral is computed to be

$$\begin{aligned} \hat{H} &= u_1^2 + u_2(u_2 - u_1) - \left(\frac{1}{2}u_1'^2 + \frac{1}{2}u_2'^2 - u_1 u_2' + \frac{1}{2}u_1^2\right) \\ &= \frac{1}{2}\left(u_1'^2 + u_2'^2 - u_1^2\right) = \text{ constant} \end{aligned}$$

These three cases do not exhaust all possible first integrals for vector case. If the functional depends only on, say $(u_1 + u_2)$, then one can hope to find new invariants for instance by changing the variables. A general method for finding the first integrals will be discussed later in Sections **??** and **??**.

Transversality and Weierstrass-Erdman conditions These conditions are quite analogous to the scalar case and their derivation is straightforward. We listen here these conditions.

The expressions $\frac{\partial F}{\partial u'_i}$, $i = 1 \ldots, n$ remain continuous at every point of an optimal trajectory, including the points where u_i is discontinuous.

If the end point b of the trajectory is unknown, the *transversality condition*

$$\hat{H}(u) = u'^T \frac{\partial F}{\partial u'} - F = 0, \quad x = b$$

is satisfied.

5.2 Legendre condition

This condition is derived similarly to the Legendre condition for scalar case. It says that the Hesian (matrix of second derivatives) of F with respect to vector u'

$$\mathcal{H}(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial u_1'^2} & \frac{\partial^2 F}{\partial u_1 \partial u'_2} \cdots & \frac{\partial^2 F}{\partial u_1 \partial u'_n} \\ \frac{\partial^2 F}{\partial u_1 \partial u'_2} & \frac{\partial^2 F}{\partial u_3'^2} \cdots & \frac{\partial^2 F}{\partial u_2 \partial u'_n} \\ \frac{\partial^2 F}{\partial u_1 \partial u'_n} & \frac{\partial^2 F}{\partial u_1 \partial u'_n} \cdots & \frac{\partial^2 F}{\partial u_n'^2} \end{pmatrix}$$

is positively defined,

$$v^T \mathcal{H} v > 0, \quad \forall V \in R_n \tag{42}$$

5.3 Variational boundary conditions

The variational condition (38) which we rewrite here for convenience

$$\left. \frac{\partial F}{\partial u_1'} \delta u_1 + \ldots + \frac{\partial F}{\partial u_n'} \delta u_n \right|_{x=a}^{x=b} = 0 \tag{43}$$

produces $2n$ boundary conditions for the Euler equations (37). If the value of a minimizer is prescribed at one of the end points, $u_i(a) = u_i^a$ then the corresponding term equation (43) is zero. If this value is not not known, the variations $\delta u_i(a)$ is free and the natural boundary condition holds, $\frac{\partial F}{\partial u_i'} = 0, \ x = a$. One of these two conditions holds for every term in (43)

$$\text{Either } \left. \frac{\partial F}{\partial u_i'} \right|_{x=a,b} = 0 \quad \text{or } \delta u_i|_{x=a,b} = 0 \quad i = 1, \ldots n. \tag{44}$$

The total number of the conditions at each endpoint is n. The missing main boundary conditions are supplemented by the natural conditions that express the requirement of optimality of the trajectory. This number agrees with the number of boundary conditions needed to solve the boundary value problem for the Euler equation for a vector minimizer.

In a general case, p relations ($p < 2n$) between boundary values of u are prescribed,

$$\beta_k(u_1(a), \ldots, u_n(a), u_1(b), \ldots, u_n(b),) = 0 \tag{45}$$

at the end points $x = a$ and $x = b$. In this case, we need to find $2n - p$ supplementary variational constraints at these points that together with (45) give $2n$ boundary conditions for the Euler equation (38) of the order $2n$.

For simplicity of notation, we introduce a $2n$ vectors w

$$\begin{array}{ll} w_k = u_k(a) & \text{if } k = 1, \ldots, n \\ w_k = u_{k-n}(b) & \text{if } k = n+1, \ldots, 2n. \end{array}$$

and

$$\begin{array}{ll} R_k = \frac{\partial F}{\partial u_k} & \text{if } k = 1, \ldots, n \\ R_k = \frac{\partial F}{\partial u_{k-n}} & \text{if } k = n+1, \ldots, 2n; \end{array}$$

equation (45) takes the form

$$\sum_{k=1}^{2n} R_k \delta w_k = 0 \tag{46}$$

The condition (46) are satisfied at the stationary and perturbed trajectories,

$$\beta_k(w + \delta w) = 0, \quad \beta_k(w + \delta w) = 0$$

therefore the $2n$ vector of variations δw_i is constrained by p linear equations

$$\sum_{i=1}^{2n} \frac{\partial \beta_k}{\partial w_i} \delta w_i = 0, \quad k = 1\ldots, p$$

One can express p variations though the remaining $2n - p$

$$\delta w_i = \sum_{k=2n-p}^{2n} q_{ik} \delta w_k, \quad i = 1, \ldots, p$$

substitute them into (46) and obtain

$$\sum_{k=2n-p}^{p} \left(R_k + \sum_{i=1}^{p} g_{ki} R_i \right) \delta w_k = 0$$

Because the variations δw_k are arbitrary, their coefficients must be zero for stationarity, which gives the variational conditions

$$R_k + \sum_{i=1}^{p} g_{ki} R_i \quad k = 2n - p, \ldots, 2n$$

This representation provides the $2n - p$ missing boundary conditions.

Example 5.2 Consider again the variational problem with the Lagrangian (40) assuming that the following boundary conditions are prescribed

$$u_1(a) = 1, \quad \beta(u_1(b), u_2(b)) = u_1^2(b) + u_2^2(b) = 1$$

Find the complementary variational boundary conditions. At the point $x = a$, the variation δu_1 is zero, and δu_2 is arbitrary, we obtain

$$\left. \frac{\partial F}{\partial u_2'} \right|_{x=a} = u_2'(a) - u_1(a) = 0$$

Since the condition $u_1(a) = 1$ is prescribed, the variational condition becomes

$$u_2'(a) = 1$$

At the point $x = b$, the variations δu_1 and δu_2 are connected by the relation

$$\frac{\partial \beta}{\partial u_1} \delta u_1 + \frac{\partial \beta}{\partial u_2} \delta u_2 = 2 u_1 \delta u_1 + 2 u_2 \delta u_2 = 0$$

which implies the representation

$$\delta u_1 = -u_2 v, \quad \delta u_2 = u_1 v$$

where v is an arbitrary scalar. The variational condition at $x = b$ becomes

$$\left(-\frac{\partial F}{\partial u_1'}u_2 + \frac{\partial F}{\partial u_2'}u_1\right)_{x=b} v = (-u_1'u_2 + (u_2' - u_1)u_1)_{x=b}\, v = 0 \quad \forall v$$

or

$$-u_1'u_2 + u_1u_2' - u_1^2\big|_{x=b} = 0.$$

We end up with four boundary conditions for the system of two differential equations of second order:

$$\begin{array}{ll} u_1(a) = 1, & u_1^2(b) + u_2^2(b) = 1, \\ u_2'(a) = 1, & u_1(b)u_2'(b) - u_1(b)'u_2(b) - u_1(b)^2 = 0. \end{array}$$

The conditions in the second row are the variational conditions.

Periodic boundary conditions Consider a variational problem with periodic boundary conditions $u(a) = u(b)$. The variational boundary conditions are obtained from the expression (43) of the variation of the functional when we use the equalities $\delta u(a) = \delta u(b)$. These conditions have the form

$$\left.\frac{\partial F}{\partial u'}\right|_{x=a} = \left.\frac{\partial F}{\partial u'}\right|_{x=b}$$

6 Lagrangian dependent on higher derivatives

Consider a more general type variational problem with the Lagrangian that depends on the minimizer and its first and second derivatives,

$$J = \int_a^b F(x, u, u', u'')dx$$

The Euler equation is derived similarly to the previous canonical case by expanding $F(x, u, u', u'')$ in Taylor series and keeping the linear terms:

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u''\right) dx$$

Integrating by parts the second term, and integrating by parts the third term twice, we obtain

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial u''}\right)\delta u\, dx + \left[\frac{\partial F}{\partial u'}\delta u + \frac{\partial F}{\partial u''}\delta u' - \frac{d}{dx}\frac{\partial F}{\partial u''}\delta u\right]_{x=a}^{x=b} \tag{47}$$

The stationarity condition (Euler equation) becomes the fourth-order differential equation

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial u''} - \frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{\partial F}{\partial u} = 0. \tag{48}$$

It is supplemented by two natural boundary conditions on each end,

$$\delta u' \frac{\partial F}{\partial u''} = 0, \quad \delta u \left[\frac{\partial F}{\partial u'} - \frac{d}{dx} \frac{\partial F}{\partial u''} \right] = 0 \quad \text{at } x = a \text{ and } x = b \tag{49}$$

or by the correspondent main conditions posed on the minimizer u and its derivative u' at the end points.

Example 6.1 The equilibrium of an elastic bending beam corresponds to the solution of the variational problem

$$\min_{u(x)} \int_0^L (\frac{1}{2}(E(x)u'')^2 - q(x)w)dx \tag{50}$$

where $u(x)$ is the deflection of the point x of the beam, $E(x)$ is the elastic stiffness of the material that can vary with x, $q(x)$ is the load that bends the beam. Any of the following kinematic boundary conditions can be considered at each end of the beam. For example, and the end $x = a$ the conditions may looks as follows
(1) A clamped end: $u(a) = 0, \quad u'(a) = 0$
(2) a simply supported end $u(a) = 0$.
(3) a free end (no kinematic conditions).

Let us find an equation for equilibrium and the missing boundary conditions in the second and third case. The Euler equation (48) becomes

$$(Ew'')'' - q = 0 \quad \in (a, b)$$

The equations (49) become

$$\delta u'(Eu'') = 0, \quad \delta u \left((Eu'')'\right) = 0$$

In the case (1) two boundary conditions are given. In the case (2) (simply supported end), the complementary variational boundary condition is $Eu'' = 0$; it expresses vanishing of the bending momentum at the simply supported end. In the case (3), the variational conditions are $Eu'' = 0$ and $(Ew'')' = 0$; the last one expresses vanishing of the bending force at the free end (the bending momentum vanishes here as well).

Generalization The Lagrangian

$$F\left(x, u, u', \ldots, u^{(k)}\right)$$

dependent on higher derivatives of u is considered similarly. The stationary condition is the $2k$-order differential equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} + \ldots + (-1)^k \frac{d^k}{dx^k} \frac{\partial F}{\partial u^{(k)}} = 0$$

supplemented at each end $x = a$ and $x = b$ of the trajectory by k boundary conditions

$$
\begin{aligned}
&\left[\frac{\partial F}{\partial u^{(k)}}\right] \delta u^{(k-1)}|_{x=a,b} = 0 \\
&\left[\frac{\partial F}{\partial u^{(k-1)}} - \frac{d}{dx}\frac{\partial F}{\partial u^{(k)}}\right] \delta u^{(k-2)}|_{x=a,b} = 0 \\
&\dots \\
&\left[\frac{\partial F}{\partial u'} - \frac{d}{dx}\frac{\partial F}{\partial u''} + \ldots + (-1)^k \frac{d^{(k-1)}}{dx^{(k-1)}}\frac{\partial F}{\partial u^{(k)}}\right] \delta u|_{x=a,b} = 0
\end{aligned}
$$

u is a vector minimizer, this vector replaces u, but the structure of the necessary conditions stays the same.