Duality, Dual Variational Principles

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Contents

1 Duality

1.1 Legendre and Young-Fenchel transforms

Duality in calculus of variation is closely related to the duality in the theory of convex function; both use the same algebraic means to pass to the dual representation. Here we review the Legendre and Young-Fenchel transforms that serve to compute the dual Lagrangian. Namely, the minimization problem (**??**) in (**??**) is an algebraic one

$$F^D(x,p,p') = \min_{u,v}\left(F(u,v) - [u,v]\cdot[p,p']\right) \tag{1}$$

In this problem, we view arguments p and p' as independent variables. This special algebraic transform is studied in convex analysis, it is called Young-Fenchel transform. If F is convex, the transform is called Legendre transform.

Definition 1.1 Let us define $F^*(\boldsymbol{z}^*)$–the *conjugate* to the $F(z)$–by the relation

$$F^*(\boldsymbol{z}^*) = \max_{\boldsymbol{z}} \{\boldsymbol{z}^* \, \boldsymbol{z} - F(\boldsymbol{z})\}, \tag{2}$$

which implies that $\boldsymbol{z}^*$ is an analog of p (compare with (12)).

Geometric interpretation Consider graph of a convex function $y = f(z)$. Assume that a straight line $az+b$ touches approaching it from below moving up (that is, increasing b). When the line touches the graph, it becomes a tangent line to a $f(z)$, that is

$$a = f'(z_0), \quad y - f(z_0) = f'(z_0)(z - z_0)$$

for some z_0. The intersection of the tangent $y = az + b$ with the axis OY occurs in the point $y = b$, or

$$y(0) = f(z_0) - f'(z_0)z_0$$

Comparing with (2), we conclude that $F(z^*) = -y(0)$ Then change the angle and repeat the experiment for all angles $a = f'$ that is for all $a \in R$.

The relation $-b(a)$ between b and a is called the conjugate or the Young-Fenchel transform of the original function $f(x)$.

Example 1.1 (Find a conjugate) Assume that $f(z) = \frac{1}{p} z^p$ We compute the conjugate:

$$z^* = f'(z) = z^{p-1}, \quad z = (z^*)^{\frac{1}{p-1}},$$

and

$$f^*(z^*) = z\, z^* - f(z) = (z^*)^{1+\frac{1}{p-1}} - \frac{1}{p} (z^*)^{\frac{p}{p-1}} = \frac{1}{q} (z^*)^q$$

where $q = \frac{p}{p-1}$ (notice that $\frac{1}{p} + \frac{1}{q} = 1$).

Young–Fenchel transform The described geometric procedure does not require dirrecetiability of convexity of $f(z)$. It is called Young–Fenchel transform and it is defined and finite for a larger class of non-differentiable functions, namely, for any Lagrangian that grows not slower than an affine function:

$$L(\boldsymbol{z}) \geq c_1 + c_2 \|\boldsymbol{z}\| \quad \forall \boldsymbol{z}, \tag{3}$$

where c_1 and $c_2 > 0$ are constants.

Example 1.2 (Find a conjugate) Consider

$$F(x) = |x|. \tag{4}$$

From (2) we have

$$F^*(x^*) = \begin{cases} 0 & \text{if} \quad |x^*| < 1, \\ \infty & \text{if} \quad |x^*| > 1. \end{cases} \tag{5}$$

Example 1.3 (Find a conjugate) Consider

$$F(x) = \exp(|x|). \tag{6}$$

From (2) we have

$$F^*(x^*) = \begin{cases} (|x^*|(\log|x^*| - 1) & \text{if} \quad |x^*| > 1, \\ \quad 0 & \text{if} \quad |x^*| > 1. \end{cases} \tag{7}$$

Example 1.4 (Find a conjugate)

$$F(x) = \begin{cases} \frac{1}{2}(|x| - 1)^2 & \text{if} \quad |x| \geq 1 \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}. \tag{8}$$

We compute

$$F^*(x^*) = \frac{1}{2}(|x^*| + 1)^2 - \frac{1}{2} \tag{9}$$

Multivariable example?

Observe that a corner point corresponds to a straight interval and vice versa. Nonconvex parts of the graph of $F(x)$ do not affect the conjugate.

Conjugate of a nonconvex function

$$f(x) = \frac{1}{2}\min\left\{(x-a)^2,\, (x+a)^2\right\}, \quad a > 0$$

is

$$f^*(x^*) = \frac{1}{2}(x^*)^2 + a|x^*|$$

If the function $f(x)$ is nonconvex growths slower than a linear, the dual may be found by considering the limit of the dual functions to the family

$$g(x, R) = \begin{cases} f(x), & \text{if} \quad |x| \leq R \\ +\infty & \text{if} \quad |x| > R \end{cases}$$

For example

$$f(x) = \sqrt{|x|} + a$$

is

$$f^*(x^*) = a$$

Algebraic features of the Legendre transform

1. The conjugate of $g = af(x) + b$ is $g^* = f^*\left(\frac{x^*-b}{a}\right)$
2. The conjugate of $g(x) = f(a\,x)$ is $g^* = f^*\left(\frac{x^*}{a}\right)$
3. The conjugate of $g(x+a) = f(a\,x)$ is $g^* = f^*\,(x^*) - a\,x^*$
4. The conjugate of $g(x) = f^{-1}(x)$ is $g^* = -x^* f^*\left(\frac{1}{a}\right)$

1.2 Second conjugate and convexification

It is easy to estimate minimum of a function from above:

$$f(x_a) \geq \min_x f(x)$$

where x_a is any value of an argument. The lower estimate is much more difficult. Duality can be used to estimate the minimum from below. The inequality

$$x\, x^* \leq f(x) + f^*(x^*), \quad \forall x, x^*$$

provides the lower estimate:

$$f(x) \geq x\, x^* - f^*(x^*) \quad \forall x, \forall x^*$$

Choosing a trial value x^* we find the lower bound.

Second conjugate We can compute the conjugate to $F^*(\boldsymbol{z}^*)$, called the *second conjugate* F^{**} to F,

$$F^{**}(\boldsymbol{z}) = \max_{\boldsymbol{z}^*} \{\boldsymbol{z}^* \cdot \boldsymbol{z} - F^*(\boldsymbol{z}^*)\}\,. \tag{10}$$

We denote the argument of F^{**} by $\boldsymbol{z}$.

If $F(\boldsymbol{z})$ is convex, then the transform is an involution.

$$F^{**}(\boldsymbol{z}) = F(\boldsymbol{z}) \quad \forall \text{ convex } F$$

If $F(\boldsymbol{z})$ is not convex, the second conjugate is the convex envelope of F (see [**?**]):

$$F^{**} = \mathcal{C}F. \tag{11}$$

The convex envelope of F is the maximal of the convex functions that does not surpass F.

1.3 Hamiltonian as a dual transform of Lagrangian

The classical version of the duality relations is based on the Legendre transform of the Lagrangian. Consider the Lagrangian $L(x, u, u')$ that is convex with respect to u'. Consider an extremal problem

$$\max_{u'} \{p\, u' - L(x, u, u')\} \tag{12}$$

that has a solution satisfying the following equation:

$$p = \frac{\partial L}{\partial u'}. \tag{13}$$

The variable p is called the *dual* or *conjugate* to the "prime" variable u; p is also called the *impulse*. Equation (13) is solvable for u', because $L(., ., u')$ is convex. We have

$$u' = \phi(p, u, x). \tag{14}$$

These relations allow us to construct the Hamiltonian H of the system.

Definition 1.2 The *Hamiltonian* is the following function of u, p, and x:

$$H(x,u,p) = p\,\phi(p,u,x) - L(x,u,\phi(p,u,x)). \tag{15}$$

The Euler equations and the dual relations yield to exceptionally symmetric representations, called *canonical equations*

$$u' = -\frac{\partial H}{\partial p}, \quad p' = \frac{\partial H}{\partial u}. \tag{16}$$

Generally, u and p are n-dimensional vectors. The canonical relations are given by $2n$ first-order differential equations for two n-dimensional vectors u and p.

The dual form of the Lagrangian can be obtained from the Hamiltonian when the variable u is expressed as a function of p and p' and excluded from the Hamiltonian. The dual equations for the extremal can be obtained from the canonical system if it is reduced to a system of n second-order differential equations for p.

Example 1.5 (Quadratic Lagrangian) Find a conjugate to the Lagrangian

$$F(u,u') = \frac{1}{2}\sigma(u')^2 + \frac{\gamma}{2}u^2. \tag{17}$$

The impulse p is

$$p = \frac{\partial F}{\partial u'} = \sigma u'.$$

Derivative u' is expressed through p as

$$u' = \frac{p}{\sigma}.$$

The Hamiltonian H is

$$H = \frac{1}{2}\frac{p^2}{\sigma} - \gamma u^2.$$

The canonical system is

$$u' = \frac{p}{\sigma}, \quad p' = \gamma u,$$

and the dual form F^* of the Lagrangian is obtained from the Hamiltonian using canonical equations to exclude u, as follows:

$$F^*(p,p') = \frac{1}{2}\left(\frac{p^2}{\sigma} - \frac{1}{\gamma}(p')^2\right).$$

The Legendre transform is an involution: The variable dual to the variable $\boldsymbol{p}$ is equal to $\boldsymbol{u}$.

2 Several variables and Duality

2.1 Several potentials

The next generalization is quite straightforward. Assume that Lagrangian depends on several potentials $\boldsymbol{u} = (u_1, \ldots, u_n)$ and on their derivatives: $n \times d$ matrix

$$\nabla \boldsymbol{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \ldots & \frac{\partial u_n}{\partial x_1} \\ \ldots & \ldots & \ldots \\ \frac{\partial u_1}{\partial x_d} & \ldots & \frac{\partial u_n}{\partial x_d} \end{pmatrix} :$$

as follows: $F = F(x, u, \nabla \boldsymbol{u})$. The Euler equation is derived by variation of each potential independently of the others. The stationary conditions become a system of n second-order equations for n unknown potentials. It has the same form as the Euler equations for the scalar case: scalar operations are simply replaced by vectorial ones and vectorial operations become matrix ones:

$$\nabla \cdot \frac{\partial F}{\partial (\nabla \boldsymbol{u})} - \frac{\partial F}{\partial \boldsymbol{u}} = 0$$

The vector form is

$$\nabla \cdot \frac{\partial F}{\partial (\nabla u_i)} - \frac{\partial F}{\partial u_i} = 0, \quad i = 1, \ldots, n$$

and the coordinate form is

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \frac{\partial F}{\partial (\frac{\partial u_i}{\partial x_k})} - \frac{\partial F}{\partial u_i} = 0, \quad i = 1, \ldots, n$$

Obviously this system degenerates into (27) when $n = 1$ and into (27) when $d = 1$.

The natural boundary conditions are

Elliptic system for two potentials

$$F = \frac{1}{2} \nabla u_1 \cdot A_1 \nabla u_1 + \nabla u_1 \cdot A_{12} \nabla u_2 + \frac{1}{2} \nabla u_2 \cdot A_2 \nabla u_2 + \phi(u_1, u_2)$$

where A_1 and A_2 are positive symmetric matrices, and the whole system is positive:

$$\det \begin{pmatrix} A_1 & A_{12} \\ A_{12} & A_2 \end{pmatrix} > 0$$

The Euler equations are

$$\nabla \cdot (A_1 \nabla u_1 + A_{12} \nabla u_2) - \frac{\partial \phi}{\partial u_1} = 0$$

$$\nabla \cdot \left(A_{12}^T \nabla u_1 + A_2 \nabla u_2 \right) - \frac{\partial \phi}{\partial u_2} = 0$$

They describe diffusion of two groups of particles that may transform to each other like groups of fast and slow neutrons in a nuclear reactor model. ϕ is a recombination term.

2.2 Dependence on Curl or divergence

Dependence on Curl Consider the simplest variational problem with Lagrangian $F(x, u, \nabla \times u)$ where u is a vector minimizer.

$$\min_u I \quad \int_\Omega F(x, u, \nabla \times u) dx + \int_{\partial\Omega} f(s, u) ds$$

We derive Euler equation is a standard manner. Consider the linearized increment of I:

$$I(u + \delta u) - I(u) = \int_\Omega \left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla \times (\delta u) \right) dx + o(\|\delta u\|)$$

Integrate by parts the last term under integral in the right-hand side of the previous expression:

$$\int_\Omega \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla \times (\delta u) dx = - \int_\Omega \nabla \times \frac{\partial F}{\partial(\nabla \times u)} \cdot \delta u dx + \int_{\partial\Omega} \frac{\partial F}{\partial(\nabla \times u)} \times n(\delta u) ds$$

Euler equation is

$$\nabla \times \frac{\partial F}{\partial(\nabla \times u)} - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega \tag{18}$$

and natural boundary condition is

$$\frac{\partial F}{\partial(\nabla \times u)} \times n + \frac{\partial f}{\partial u} = 0$$

Dependence on Divergence Consider the simplest variational problem with Lagrangian $F(x, \boldsymbol{u}, \nabla \cdot \boldsymbol{u})$ where $\boldsymbol{u}$ is a vector minimizer.

$$\min_{\boldsymbol{u}} I \quad \int_\Omega F(x, \boldsymbol{u}, \nabla \cdot \boldsymbol{u}) dx + \int_{\partial\Omega} f(s, \boldsymbol{u}) ds$$

We derive Euler equation is a standard manner. It has the form:

$$\nabla \left(\frac{\partial F}{\partial(\nabla \cdot \boldsymbol{u})} \right) - \frac{\partial F}{\partial \boldsymbol{u}} = 0 \quad \text{in } \Omega$$

and natural boundary condition is

$$\frac{\partial F}{\partial(\nabla \cdot \boldsymbol{u})} n + \frac{\partial f}{\partial \boldsymbol{u}} = 0$$

2.3 Projection approach

Types of variables Let us discuss differential constraints on the vector $\boldsymbol{v}$. Usually, $\boldsymbol{v}$ consists of either curlfree or divergencefree vectors. They correspond to some potentials and are subject to linear differential constraints that express the integrability. We distinguish three types of minimizers $\boldsymbol{v}$ to the variational problem:

1. Potentials: Differentiable variables $\boldsymbol{u}$, such as temperature or displacement. The L_p norm of all partial derivatives of $\boldsymbol{w}$ is bounded. The Lagrangian L explicitly depends on these variables and on some partial derivatives $\frac{\partial u_k}{\partial x_j}$ of them, $L(\boldsymbol{w}, \nabla \boldsymbol{w})$.

2. Nondifferentiable variables $\boldsymbol{f}$, such as the density of the sources. These variables belong to L_p spaces, and their partial derivatives are not included in the Lagrangian.

3. Variables $\boldsymbol{v}$ that represent either curlfree or divergencefree vectors, such as currents or electrical fields. The differential operators curl, divergence, strain, depend on certain combinations of partial derivatives $\frac{\partial u_k}{\partial x_j}$ or on matrix ∇u They can be viewed as a projection (linear mapping) of $n \times d$ matrix ∇u onto a p dimensional space $\mathcal{V}$ of variables $v \in \mathcal{V}$:

   $$v_i = \sum_j^n \sum_k^d a_{ijk} \frac{\partial u_k}{\partial x_j}$$

   Notice that projection is defined by three indices: the first shows the coordinate of the resulting vector v and the second and third point on the coordinates of the matrix component of the argument. Such arrays $A = \{a_{ijk}\}$ are called third-rank tensors. The differential form can be rewritten as

   $$v = A \cdot \nabla \cdot u$$

   In many problem, it is convenient to consider the general projection formula instead of listing of special cases. This type of variable appears only in multidimensional variational problems. They are the focus of our consideration.

Here, we derive Euler-Lagrange equation for the general case. Then we consider examples of the projection tensor $A = \{a_{ijk}\}$.

Derivation of the Euler-Lagrange equation Assume that the variational problem takes the form

$$\min_u I \quad I = \int_\Omega F(x, u, v)dx + \int_{\partial\Omega} f(s, u)ds, \quad v = A \cdot \nabla \cdot u$$

Using Lagrange multipliers vector λ, we rewriter I as

$$I = \int_\Omega \left[F(u, v) + \lambda(\cdot A \cdot \nabla \cdot u - v)\right] dx + \int_{\partial\Omega} f(s, u)ds$$

Performing variations with respect to v and u, we obtain

$$\delta v : \quad \frac{\partial F}{\partial v} - \lambda = 0 \quad \text{in } \Omega \tag{19}$$

and

$$\delta u: \quad I_u = \int_\Omega \left[\frac{\partial F}{\partial u}\delta u + \underline{\lambda \cdot A \cdot \nabla \cdot \delta u} \right] dx + \int_{\partial\Omega} \frac{\partial f}{\partial u} ds = 0 \tag{20}$$

Integration by parts of the underlined term $\underline{I}$ of I_u leads to:

$$\underline{I} = \int_\Omega \lambda \cdot A \cdot \nabla \cdot \delta u dx = \int_\Omega \delta u \cdot [-A^* \cdot \nabla \cdot \lambda] \, dx + \int_{\partial\Omega} [\lambda \cdot A \cdot n] \cdot \delta u ds$$

where $A*$ is the third-rank tensor with coefficients

$$a^*_{ijk} = a_{kji}$$

Substituting the result into (20) and noticing that δu is a free variation, we obtain the conditions in the domain

$$\delta u: \quad -A^* \cdot \nabla \cdot \lambda + \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega \tag{21}$$

and on the boundary

$$\lambda \cdot A \cdot n + \frac{\partial f}{\partial u} = 0 \quad \text{on } \partial\Omega \tag{22}$$

Finally, we exclude λ from (19) and (22). In order to do it, we multiply the equality in (19) by $A^* \cdot \nabla \cdot$ from the left, take (21) into account, and obtain Euler-Lagrange equations:

$$A^* \cdot \nabla \cdot \frac{\partial F}{\partial v} - \frac{\partial F}{\partial u} = 0, \quad v = A \cdot \nabla \cdot u \quad \text{in } \Omega$$

and boundary terms

$$\left[\frac{\partial F}{\partial v} \cdot A \cdot n + \frac{\partial f}{\partial u} \right] \cdot \delta u = 0 \quad \text{on } \partial\Omega$$

Examples

Example 2.1 Assume that a Lagrangian depends on divergence $v = \nabla \cdot \boldsymbol{u}$ or

$$\sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} = v, \tag{23}$$

In this case the constraints (**??**) are set as:

$$r = 1, \quad g_1 = q, \quad a_{ijk} = \delta_{1i}\delta_{ik}, \tag{24}$$

where δ_{ab} is the Kronecker symbol; $\mathcal{A}$ is the $(1 \times n \times d)$ tensor.

The tensor operator $A^* \cdot \nabla \cdot$ becomes gradient; The Euler equations are:

$$\nabla \frac{\partial F}{\partial v} - \frac{\partial F}{\partial \boldsymbol{u}} = 0$$

Example 2.2 Consider a three-dimensional solenoidal field $\boldsymbol{v} = \nabla \times \boldsymbol{u}$

$$\frac{\partial e_3}{\partial x_2} - \frac{\partial e_2}{\partial x_3} = v_1, \quad \frac{\partial e_1}{\partial x_3} - \frac{\partial e_3}{\partial x_1} = v_2, \quad \frac{\partial e_2}{\partial x_1} - \frac{\partial e_1}{\partial x_2} = v_3.$$

To rewrite the constraints in the tensor form we set the $3 \times 3 \times 3$ tensor $\mathcal{A} = \{a_{ijk}\}$ equal to the Levi-Civita tensor. It has the following nonzero elements:

$$a_{132} = a_{213} = a_{321} = 1, \quad a_{123} = a_{231} = a_{312} = -1.$$

In this case, $A \cdot \nabla \cdot = \nabla \times$, and $A^* \cdot \nabla \cdot = \nabla \times$; the Euler equation (**??**) follows.

Remark 2.1 The form (**??**) of differential constraints was used for classification of the variational problems starting from [**?**, **?**, **?**]. This form is convenient, but not imperative. At long last, all the fields are linear combinations of elements of gradients of some potentials, because they are linear combinations of some partial derivatives. Therefore, the classical form $L(\boldsymbol{x}, \boldsymbol{w}, \nabla \boldsymbol{w})$ of Lagrangian is equivalent to (**??**).

2.4 Dual form

Projection approach is applied to find dual variational principles. Instead of excluding λ from (19) and (22), we solve these equations for $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\boldsymbol{u} = \phi(\lambda, \mu), \quad \boldsymbol{v} = \psi(\lambda, \mu),$$

where

$$\mu = \mathcal{A} \cdot \nabla \cdot \lambda \tag{25}$$

and substitute these into expressions for extended Lagrangian. In other words, we perform the Legendre transform of the Lagrangian $L(\boldsymbol{u}, \boldsymbol{v})$. The obtained dual Lagrangian

$$F^D(x, \lambda, \mu) = F(x, \phi(\lambda, \mu), \psi(\lambda, \mu)) - \lambda \cdot \psi(\lambda, \mu) - \mu \cdot \phi(\lambda, \mu)$$

corresponds to the dual variational problem.

$$I^D = \max_{\lambda, \mu \text{ as in } (25)} \int_\Omega F^D(x, \lambda, \mu) dx$$

Examples

$$F = \frac{C}{2} (\nabla \times \boldsymbol{u})^2 + \frac{\gamma}{2} (\boldsymbol{u})^2 \tag{26}$$

We introduce $\boldsymbol{\lambda}$ to account for the differential constraint and pass to the expended Lagrangian

$$L = \frac{C}{2} (\boldsymbol{v})^2 + \frac{\gamma}{2} (\boldsymbol{u})^2 + \boldsymbol{\lambda} \cdot (\nabla \times \boldsymbol{u} - \boldsymbol{v})$$

Optimality conditions are

$$C\boldsymbol{v} = \boldsymbol{\lambda} \quad \text{and } \gamma \boldsymbol{u} - \nabla \times \boldsymbol{\lambda} = 0$$

From these two equations, we express $\boldsymbol{v}$ and $\boldsymbol{u}$ as functions of $\boldsymbol{\lambda}$ and $\nabla \times \boldsymbol{\lambda}$:

$$\boldsymbol{u} = \frac{1}{\gamma} \nabla \times \boldsymbol{\lambda}, \quad \boldsymbol{v} = \frac{1}{C} \boldsymbol{\lambda},$$

and substitute them into (28). The result

$$F^D(\boldsymbol{\lambda}) = \frac{1}{2C} (\boldsymbol{\lambda})^2 + \frac{1}{2\gamma} (\nabla \times \boldsymbol{\lambda})^2$$

is the dual form of the variational problem.

Example 2.3 (Pile of sand) Consider the Lagrangian $L = |\nabla u| + a\, u$ and find a dual form of it. We rewite the Lagrangian as

$$L = \max_{\boldsymbol{\lambda}} [|\boldsymbol{v}| + \boldsymbol{\lambda}(\nabla u - \boldsymbol{v}|) + a\, u]$$

The stationary conditions with respect to u and $\boldsymbol{v}$ are

$$\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - \boldsymbol{\lambda} = 0 \quad \nabla \cdot \boldsymbol{\lambda} - a = 0$$

and the dual system is:

$$|\boldsymbol{\lambda}| = 1 \quad \nabla \cdot \boldsymbol{\lambda} = a \tag{27}$$

Example 2

$$F = \frac{C}{p} (\nabla u)^p + \frac{\gamma}{2} u^2 + a\, u, \quad p > 1 \tag{28}$$

We introduce vector $\boldsymbol{\lambda}$ to account for the differential constraint and pass to the expended Lagrangian

$$L = \frac{C}{p} (\boldsymbol{v})^p + \frac{\gamma}{2} (\boldsymbol{u})^2 + a\, u + \boldsymbol{\lambda} \cdot (\nabla u - \boldsymbol{v})$$

Optimality conditions are

$$C\boldsymbol{v}^{p-1} = \boldsymbol{\lambda} \quad \text{and } \gamma \boldsymbol{u} + a - \nabla \cdot \boldsymbol{\lambda} = 0$$

From these two equations, we express $\boldsymbol{v}$ and $\boldsymbol{u}$ as functions of $\boldsymbol{\lambda}$

$$\boldsymbol{u} = \frac{1}{\gamma} (\nabla \cdot \boldsymbol{\lambda} - a), \quad \boldsymbol{v} = \left(\frac{\boldsymbol{\lambda}}{C} \right)^{\frac{1}{p-1}},$$

and substitute them into (28). The result

$$F^D(\boldsymbol{\lambda}) = C^{-\frac{1}{p-1}} (\boldsymbol{\lambda})^{\frac{p}{p-1}} + \frac{1}{2\gamma} (\nabla \cdot \boldsymbol{\lambda} - a)^2$$

is the dual form of the variational problem. We rewrite it as:

$$F^D(\boldsymbol{\lambda}) = \frac{1}{2\gamma} (\nabla \cdot \boldsymbol{\lambda})^2 - \frac{a}{\gamma} \nabla \cdot \boldsymbol{\lambda} + \frac{1}{C^{\frac{1}{p-1}}} (\boldsymbol{\lambda})^{\frac{p}{p-1}} + a^2$$

2.5 Lower bound

The duality is very important tool because it provide the means to establish lower bound of the variational problem. The upper bound of it is easy: every trial function $\boldsymbol{u}_{trial}$ provide such a bound. To find the lower bound, we use the dual form:

$$I = \max_{\boldsymbol{\lambda}} \int_\Omega F^D(x, \boldsymbol{\lambda}, A^* \cdot \nabla \cdot \boldsymbol{\lambda}) dx$$

This relation implies that any trial function $\boldsymbol{\lambda}$ correspond to the lower bound of the functional. The difference of the upper and lower bound provide the measure of the preciseness of both approximations: The inequalities

$$\int_\Omega F^D(x, \boldsymbol{\lambda}, A^* \cdot \nabla \cdot \boldsymbol{\lambda}) dx \leq I \leq \int_\Omega F(x, \boldsymbol{u}, A \cdot \nabla \cdot \boldsymbol{u}) dx$$

hold for any admissible $\boldsymbol{u}$ and $\boldsymbol{\lambda}$ that satisfy main boundary conditions.

3 Complex conductivity

3.1 Equations of Complex Conductivity

The Process Consider conductivity in a dissipative medium with inductance and capacity along with resistivity. The current $\boldsymbol{j}$ and the electric field $\boldsymbol{e}$ are now functions of time and space coordinates. The current is divergencefree, and the field is curlfree:

$$\nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0. \tag{29}$$

These constraints allow us to introduce a vector potential $\boldsymbol{a}$ of the current field $\boldsymbol{j}$ and a scalar potential ϕ of the electrical field $\boldsymbol{e}$ through the relations

$$\boldsymbol{j} = \nabla \times \boldsymbol{a}, \quad \boldsymbol{e} = -\nabla \phi. \tag{30}$$

Consider a body Ω occupied by a conducting material and suppose that this body is loaded on the boundary $S = \partial\Omega$. The boundary conditions are similar to those for a conducting material (see Chapter 4)

$$\phi = \phi_0 \text{ on } S_1, \quad \boldsymbol{n} \cdot \boldsymbol{j} = j_0 \text{ on } S_2, \quad S_1 \cup S_2 = S, \tag{31}$$

where $\boldsymbol{n}$ is the normal.

Assume that the properties of the material are local in space and in time: The current field and its derivatives at a point $\boldsymbol{x} \in \Omega$ at the moment t depend only on the electrical field and its derivatives at the same point at the same moment of time. Assume that the material is linear in the following sense: A linear combination of the current and its time derivatives linearly depends on a linear combination of the field and its time derivatives:

$$\sum_k a_k \frac{\partial^k \boldsymbol{j}}{\partial t^k} = \sum_k b_k \frac{\partial^k \boldsymbol{e}}{\partial t^k}. \tag{32}$$

Here $a_k = a_k(\boldsymbol{x})$ and $b_k = b_k(\boldsymbol{x})$ are some time-independent coefficients, which are scalars (for the isotropic conductors) or symmetric matrices (for the anisotropic ones). The properties of the material (i.e., the scalar or matrix parameters $a_k,\ b_k$) do not depend on time.

Monochromatic Excitation Consider steady-state oscillations in a dissipative medium caused by a monochromatic excitation. The electrical field and current in the material are also monochromatic, i.e.,

$$\begin{aligned} \boldsymbol{j}^s(\boldsymbol{x},t) &= (\boldsymbol{J}(\boldsymbol{x})e^{i\omega t})' = \boldsymbol{J}'(\boldsymbol{x})\cos\omega t + \boldsymbol{J}''(\boldsymbol{x})\sin\omega t, \\ \boldsymbol{e}^s(\boldsymbol{x},t) &= (\boldsymbol{E}(\boldsymbol{x})e^{i\omega t})' = \boldsymbol{E}'(\boldsymbol{x})\cos\omega t + \boldsymbol{E}''(\boldsymbol{x})\sin\omega t, \end{aligned}$$

where $\Phi_0(s)$, $J_0(s)$, $\boldsymbol{J}(\boldsymbol{x})$, and $\boldsymbol{E}(\boldsymbol{x})$ are the complex-valued Fourier coefficients of corresponding functions, and s is the coordinate along the boundary. Here, the real and imaginary parts of variables are denoted by the superscripts $'$ and $''$, i.e., $\boldsymbol{c} = \boldsymbol{c}' + i\boldsymbol{c}''$.

The Complex-Valued Conductivity Equations The linearity of the constitutive relations (32) leads to a linear relationship between the vectors $\boldsymbol{J}(\boldsymbol{x})$ and $\boldsymbol{E}(\boldsymbol{x})$:

$$\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E}, \tag{33}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\omega) = \boldsymbol{\sigma}'(\omega) + i\boldsymbol{\sigma}''(\omega)$ is a complex conductivity tensor that depends on the frequency of oscillations [**?**]. For an isotropic material with state law (32), the tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \frac{\sum_k (-i\omega)^k a_k}{\sum_k (-i\omega)^k b_k} I, \tag{34}$$

where I is a unit matrix.

The divergencefree nature of the current field and the curlfree nature of the electrical field means that the Fourier coefficients of these fields satisfy relations similar to (29)

$$\nabla \cdot \boldsymbol{J} = 0, \quad \nabla \times \boldsymbol{E} = 0. \tag{35}$$

Therefore, they allow the representation (see (30))

$$\boldsymbol{J} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla\Phi, \tag{36}$$

where $\boldsymbol{A}$ and Φ are the Fourier coefficients of the potentials $\boldsymbol{a}$ and ϕ.

The boundary conditions (31) lead to the relations

$$\Phi = \Phi_0 \text{ on } S_1,\ \boldsymbol{n} \cdot \boldsymbol{J} = J_0 \text{ on } S_2,\ S_1 \cup S_2 = S, \tag{37}$$

where Φ_0 and J_0 are the Fourier coefficients of the functions ϕ_0 and j_0.

A harmonic oscillation in the conducting media is described by the constitutive relations (33) and differential equations (35), (36) in conjunction with the boundary conditions (37).

The System of Real First-Order Equations The complex-valued equations (33), (35), (36), and (37) describe the conductance of the medium. They look exactly like the equations for the real conductivity; however, they correspond to more complicated processes. Indeed, the complex-valued differential equations (35) and (36) form a fourth-order system of differential equations for the real and imaginary parts of the variables $\boldsymbol{J}'$ and $\boldsymbol{E}$,

$$\nabla \cdot \boldsymbol{J}' = 0, \quad \nabla \cdot \boldsymbol{J}'' = 0, \quad \nabla \times \boldsymbol{E}' = 0, \quad \nabla \times \boldsymbol{E}'' = 0. \tag{38}$$

These equations are identically satisfied if the following potentials are introduced:

$$\boldsymbol{J}' = \nabla \times \boldsymbol{A}', \quad \boldsymbol{J}'' = \nabla \times \boldsymbol{A}'', \quad \boldsymbol{E}' = -\nabla \Phi', \quad \boldsymbol{E}'' = -\nabla \Phi''. \tag{39}$$

The currents and electrical fields are connected by the constitutive relations (33)

$$\begin{aligned} -\boldsymbol{J}' &= -\boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{\sigma}''\boldsymbol{E}'', \\ \boldsymbol{J}'' &= \boldsymbol{\sigma}''\boldsymbol{E}' + \boldsymbol{\sigma}'\boldsymbol{E}''. \end{aligned} \tag{40}$$

The vector form of the last equations is

$$\begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix} = \boldsymbol{D}_{EE} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}, \tag{41}$$

where the block-matrix

$$\boldsymbol{D}_{EE} = \begin{pmatrix} -\boldsymbol{\sigma}' & \boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}'' & \boldsymbol{\sigma}' \end{pmatrix} \tag{42}$$

is the conductivity matrix of the medium. (Recall that $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ are $d \times d$ matrices of properties)

The boundary conditions (37) can be rewritten as

$$\Phi' = \Phi_0' \text{ on } S_1, \tag{43}$$

$$\Phi'' = \Phi_0'' \text{ on } S_1, \tag{44}$$

$$\boldsymbol{n} \cdot \boldsymbol{J}' = J_0' \text{ on } S_2, \tag{45}$$

$$\boldsymbol{n} \cdot \boldsymbol{J}'' = J_0'' \text{ on } S_2. \tag{46}$$

The formulated system of the real-valued differential equations and boundary conditions describes the conductivity of the complex conducting medium. Notice that it has double dimensions compared to the real conductivity problem.

The conductivity is defined by two tensors $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}'$. The real part $\boldsymbol{\sigma}'$ is nonnegative,

$$\boldsymbol{\sigma}' \geq 0, \tag{47}$$

because the dissipation rate is nonnegative. Indeed, the energy dissipation averaged over the period of oscillations is equal to:

$$\frac{\omega}{2\pi} \int_t^{t+\frac{2\pi}{\omega}} \boldsymbol{j}^s \cdot \boldsymbol{e}^s dt = \frac{1}{2}(\boldsymbol{J}' \cdot \boldsymbol{E}' + \boldsymbol{J}'' \cdot \boldsymbol{E}'') = \frac{1}{2}(\boldsymbol{E}' \cdot \boldsymbol{\sigma}'\boldsymbol{E}' + \boldsymbol{E}'' \cdot \boldsymbol{\sigma}'\boldsymbol{E}'') \tag{48}$$

(see [**?**]). The condition (47) expresses the positiveness of the dissipation rate.

Real Second-Order Equations The system (39), (40) of four first-order differential equations can be rewritten as a system of two second-order equations. We do it in four different ways, and we end up with four equivalent systems. Each of them turns out to be Euler–Lagrange equations for a variational problem.

First, we express the fields though scalar potentials Φ' and Φ'' and take the divergence $(\nabla\cdot)$ of the right- and left-hand sides of (40). The left-hand-side terms $\nabla \cdot \boldsymbol{j}', \nabla \cdot \boldsymbol{j}''$ vanish and we obtain:

$$\begin{aligned} 0 &= \nabla \cdot [-\boldsymbol{\sigma}' \nabla \Phi' + \boldsymbol{\sigma}'' \nabla \Phi''], \\ 0 &= \nabla \cdot [\ \boldsymbol{\sigma}'' \nabla \Phi' + \boldsymbol{\sigma}' \nabla \Phi'']. \end{aligned}$$

Thus we obtain two second-order equations for two potentials Φ' and Φ''. The vector form of this system is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla\cdot & 0 \\ 0 & \nabla\cdot \end{pmatrix} \boldsymbol{D}_{EE} \begin{pmatrix} \nabla\Phi' \\ \nabla\Phi'' \end{pmatrix}. \tag{49}$$

We may also rewrite this system of equations taking any other pair of four scalar and vector potentials (39) and excluding the other two. For example, let us exclude the fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$. First, we solve equations (40) for $\boldsymbol{E}'$ and $\boldsymbol{E}''$:

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{JJ} \begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{50}$$

where

$$\boldsymbol{D}_{JJ} = \begin{pmatrix} -(\boldsymbol{\sigma}' + \boldsymbol{\sigma}'' \boldsymbol{\sigma}'^{-1} \boldsymbol{\sigma}'')^{-1} & (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}' \boldsymbol{\sigma}''^{-1} \boldsymbol{\sigma}')^{-1} \\ (\boldsymbol{\sigma}'' + \boldsymbol{\sigma}' \boldsymbol{\sigma}''^{-1} \boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}' + \boldsymbol{\sigma}'' \boldsymbol{\sigma}'^{-1} \boldsymbol{\sigma}'')^{-1} \end{pmatrix}. \tag{51}$$

(Note that $\boldsymbol{D}_{EE} = \boldsymbol{D}_{JJ}^{-1}$.)

Take the curl $(\nabla\times)$ of the right- and left-hand sides of both equations (50). The left-hand-side terms identically vanish, and we obtain two vector equations:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla\times & 0 \\ 0 & \nabla\times \end{pmatrix} \boldsymbol{D}_{JJ} \begin{pmatrix} -\nabla \times \boldsymbol{A}' \\ \nabla \times \boldsymbol{A}'' \end{pmatrix}. \tag{52}$$

Here we use the representation (39) of current fields $\boldsymbol{J}'$ and $\boldsymbol{J}''$ through the vector potentials $\boldsymbol{A}'$ and $\boldsymbol{A}''$.

We may as well solve (40) for the fields $\boldsymbol{E}'$ and $\boldsymbol{J}''$ and obtain

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}' \end{pmatrix} = \boldsymbol{D}_{JE} \begin{pmatrix} \boldsymbol{J}'' \\ \boldsymbol{E}'' \end{pmatrix}, \tag{53}$$

where

$$\boldsymbol{D}_{JE} = \begin{pmatrix} (\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1} \boldsymbol{\sigma}'' \\ \boldsymbol{\sigma}'' (\boldsymbol{\sigma}')^{-1} & \boldsymbol{\sigma}' + \boldsymbol{\sigma}'' (\boldsymbol{\sigma}')^{-1} \boldsymbol{\sigma}'' \end{pmatrix}. \tag{54}$$

Recall that $\boldsymbol{E}'$ is curlfree and $\boldsymbol{J}''$ is divergencefree. Therefore, by using (38) and (39) we arrive at the following system of second-order equations:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla\times & 0 \\ 0 & \nabla\cdot \end{pmatrix} \boldsymbol{D}_{JE} \begin{pmatrix} \nabla \times \boldsymbol{A}' \\ \nabla\Phi'' \end{pmatrix}. \tag{55}$$

Similarly, we solve (40) for $\boldsymbol{J}'$ and $\boldsymbol{E}''$ and obtain

$$\begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix} = \boldsymbol{D}_{EJ} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}, \tag{56}$$

where

$$\boldsymbol{D}_{EJ} = \begin{pmatrix} \boldsymbol{\sigma}' + \boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' & -(\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}'' \\ -\boldsymbol{\sigma}''(\boldsymbol{\sigma}')^{-1} & (\boldsymbol{\sigma}')^{-1} \end{pmatrix}. \tag{57}$$

(Note that $\boldsymbol{D}_{JE}^{-1} = \boldsymbol{D}_{EJ}$.)

Again, the operations $(\nabla\cdot)$ and $(\nabla\times)$ eliminate the corresponding terms on the left-hand side in equations (56). Applying these operators, we obtain the second-order system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla\cdot & 0 \\ 0 & \nabla\times \end{pmatrix} \boldsymbol{D}_{JE} \begin{pmatrix} \nabla\Phi' \\ \nabla\times\boldsymbol{A}'' \end{pmatrix}. \tag{58}$$

We have written four different forms of the same equations. The systems (49), (52), (55), and (58) are equivalent to each other and to the original system (38). Each of them in conjunction with the boundary conditions (43)–(46) allows us to find the solution that describes the processes in the conducting medium. We now show that each of them represents the Euler equations for a corresponding variational problem.

3.2 Quartet of variational principles

Let us establish variational principles for the problem of complex conductivity. There is no direct complex analogue to the variational principles for the real-valued problem because the inequalities cannot be considered for complex variables. However, the real-valued differential equations just described are the stationary conditions for some real-valued functionals. These functionals lead to variational principles that describe the complex conductivity processes.

First, we formulate two minimax variational principles. They follow naturally from the equations in the form (49) and (52). Then we obtain two minimal variational principles based on the equations of the problem in the form (55) and (58). Finally, we discuss the relation between these four principles, referring to the procedure of Legendre transform.

The Minimax Variational Principle for the Fields Consider the following variational minimax problem:

$$\min_{\boldsymbol{E}''} \max_{\boldsymbol{E}'} U_{EE}, \tag{59}$$

where the fields $\boldsymbol{E}'$, $\boldsymbol{E}''$ are subject to the constraints

$$\begin{aligned} \boldsymbol{E}'' &= -\nabla\Phi'', \quad \Phi'' = \Phi_0'' \text{ on } S_1, \\ \boldsymbol{E}' &= -\nabla\Phi', \quad \Phi' = \Phi_0' \text{ on } S_1; \end{aligned}$$

the functional U_{EE} is

$$U_{EE} = \int_{\Omega} W_{EE}(\boldsymbol{E}', \boldsymbol{E}'') + \int_{S_2} [\Phi'' J_0'' - \Phi' J_0']; \tag{60}$$

and

$$W_{EE}(\boldsymbol{E}', \boldsymbol{E}'') = \frac{1}{2} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}^T \boldsymbol{D}_{EE} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{E}'' \end{pmatrix}. \tag{61}$$

The matrix $\boldsymbol{D}_{EE}$ is defined in (42).

The vanishing of the first variation with respect to $\boldsymbol{E}'$, $\boldsymbol{E}''$ of the functional U_{EE} (see (60)) leads to two Euler–Lagrange equations that coincide with (40). One can check that they coincide with the original system of equations in the form (49) and with the boundary conditions (45), (46). The boundary conditions (43), (44) must be assumed at all admissible fields.

To check the sense of optimality of the stationary solution we examine the sign of the second variation of the functional; see, for example, [**?**]. The second variation is the main term of the increment of the functional at the perturbed solution of the Euler–Lagrange equation. Whereas the first variation is zero at the solution, the second variation of the cost is proportional to the quadratic form

$$(\delta\boldsymbol{E}\,, \delta\boldsymbol{E}'')^T \boldsymbol{D}_{EE}(\delta\boldsymbol{E}\,, \delta\boldsymbol{E}'').$$

The functional has a local minimum at the stationary solution if the second variation is positive, and it has a local maximum at the stationary solution if the second variation is negative. The sign of the variation is determined by the matrix $\boldsymbol{D}_{EE}$.

Here the second variation is neither positive nor negative, because the matrix $\boldsymbol{D}_{EE}$ is neither positive nor negative definite. The stationary solution corresponds to the saddle point of the functional. The variational problem is of the minimax type.

The Minimax Variational Principle for the Currents Similarly, one can derive the Euler–Lagrange equations of the variational problem

$$\max_{\boldsymbol{J}'} \min_{\boldsymbol{J}''} U_{JJ}, \tag{62}$$

where the fields $\boldsymbol{J}', \boldsymbol{J}''$ are

$$\{\boldsymbol{J}' : \ \boldsymbol{J}' = \nabla \times \boldsymbol{A}', \quad \boldsymbol{n} \cdot \boldsymbol{J}' = J_0' \text{ on } S_2\},$$
$$\{\boldsymbol{J}'' : \ \boldsymbol{J}'' = \nabla \times \boldsymbol{A}, \quad \boldsymbol{n} \cdot \boldsymbol{J}'' = J_0'' \text{ on } S_2\};$$

the functional U_{JJ} is

$$U_{JJ} = \int_{\Omega} W_{JJ}(\boldsymbol{J}',\ \boldsymbol{J}'') + \int_{S_1} [\Phi_0''\ \boldsymbol{n} \cdot \boldsymbol{J}'' - \Phi_0'\ \boldsymbol{n} \cdot \boldsymbol{J}']; \tag{63}$$

and

$$W_{JJ}(\boldsymbol{J}',\ \boldsymbol{J}'') = \frac{1}{2} \begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix}^T \boldsymbol{D}_{JJ} \begin{pmatrix} -\boldsymbol{J}' \\ \boldsymbol{J}'' \end{pmatrix}. \tag{64}$$

The matrix $\boldsymbol{D}_{JJ}$ is defined by (51).

We check that the Euler equations for the functional (63) coincide with equations (50) that describe the same problem in different notation.

The matrix $\boldsymbol{D}_{JJ}$ is neither positive nor negative definite, hence the second variation of the functional U_{JJ} is again neither positive nor negative. We conclude that the variational problem (63) is of the minimax type.

Remark 3.1 The minimax nature of the variational principles (59) and (62) does not allow us to apply the technique developed to the bounds. This technique uses the fact that the energy (i.e., the value of the functional) on any trial field should exceed the actual energy stored in the material. Therefore the energy on any trial field provides an upper bound on the actual energy. For the minimax principles (59), (62), however, the situation is different. Consider, for example, the problem (59) and let us calculate the energy on trial fields of two potentials Φ' and Φ''. The actual energy is increased if the trial field $\nabla\Phi''$ differs from the optimal one and is decreased if the other trial field $\nabla\Phi'$ is not optimal. The value of the functional (60) on the trial fields can be lower or higher than the actual energy and cannot bound the functional (60) from either side.

The First Minimal Variational Principle Consider the following variational problem for the variables $\boldsymbol{J}'$ and $\boldsymbol{E}''$:

$$\min_{\boldsymbol{J}'} \min_{\boldsymbol{E}''} U_{JE}, \tag{65}$$

where the fields $\boldsymbol{J}', \boldsymbol{E}''$ are

$$\begin{aligned} \{\boldsymbol{J}' :\ \boldsymbol{J}' &= \nabla \times \boldsymbol{A}', \quad \boldsymbol{n} \cdot \boldsymbol{J}' = J_0' \text{ on } S_2\}, \\ \{\boldsymbol{E}'' :\ \boldsymbol{E}'' &= -\nabla\Phi'', \quad \Phi'' = \Phi_0'' \text{ on } S_1\}; \end{aligned}$$

the functional U_{JE} is

$$U_{JE} = \int_\Omega W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') - \int_{S_1} \boldsymbol{n} \cdot \boldsymbol{J}' \Phi_0' + \int_{S_2} \Phi'' J_0''; \tag{66}$$

and

$$W_{JE}(\boldsymbol{J}', \boldsymbol{E}'') = \frac{1}{2} \begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix}^T \boldsymbol{D}_{JE} \begin{pmatrix} \boldsymbol{J}' \\ \boldsymbol{E}'' \end{pmatrix}. \tag{67}$$

The matrix $\boldsymbol{D}_{JE}$ is defined in (54). The first variation of (65) with respect to $\boldsymbol{J}'$ and $\boldsymbol{E}''$ coincides with the system of original equations in the form (55) and the boundary conditions (44), (46).

Note that this time the quadratic form (67) is positive. This follows from the physically clear condition (47). As we see, this functional is equal to the whole energy dissipated in the body Ω during one period of oscillation (see (48)).

The second variation $\delta^2 U_{JE}$ of the functional (66) is positive due to the positivity of the matrix $\boldsymbol{D}_{JE}$ (for physical reasons we always suppose that $\boldsymbol{\sigma}' \geq 0$ or the dissipation rate is positive). For the quadratic functional (66) the positivity of the second variation is sufficient to guarantee the global minimum at a stationary point [**?**].

The Second Minimal Variational Principle Similarly, we consider the variational problem

$$\min_{\boldsymbol{J}''} \min_{\boldsymbol{E}'} U_{EJ}, \tag{68}$$

where the fields $\boldsymbol{J}'', \boldsymbol{E}'$ are

$$\begin{array}{llll} \{\boldsymbol{J}'' : & \boldsymbol{J}'' = \nabla \times \boldsymbol{A}'', & \boldsymbol{n} \cdot \boldsymbol{J}'' = J_0'' \text{ on } S_2\}, \\ \{\boldsymbol{E}' : & \boldsymbol{E}' = -\nabla \Phi', & \Phi' = \Phi_0' \text{ on } S_1\}; \end{array}$$

the functional U_{EJ} is:

$$U_{EJ} = \int_\Omega W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') + \int_{S_1} \boldsymbol{n} \cdot \boldsymbol{J}'' \Phi_0'' - \int_{S_2} \Phi' J_0'; \tag{69}$$

and

$$W_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') = \frac{1}{2} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}^T \boldsymbol{D}_{EJ} \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{J}'' \end{pmatrix}. \tag{70}$$

The matrix $\boldsymbol{D}_{EJ}$ is defined in (57)

In considering the first variation of the functional (68), we conclude again that the Euler equations for the functional (69) coincide with the system of original equations in the form (58) and the boundary conditions (43), (45).

One could also see that the second variation of this functional is positive if $\boldsymbol{\sigma}' \geq 0$.

Remark 3.2 Note that the two variational principles are equivalent:

$$W_{JE}(\boldsymbol{J}, \boldsymbol{E}) = W_{EJ}(\boldsymbol{E}, \boldsymbol{J}), \tag{71}$$

This feature is specific for this problem; usually we meet two different variational principles of minimization of the potential energy and the complementary energy (for example, the Dirichlet and Thomson principles).

3.3 Legendre Transform

One can check that the pairs of variational problems, (59) and (62), (65) and (68), are mutually dual [**?**]. The matrices associated with the quadratic forms, (61) and (64), and (67) and (70), are reciprocally inverse, i.e.,

$$\boldsymbol{D}_{EE} = \boldsymbol{D}_{JJ}^{-1}, \quad \boldsymbol{D}_{EJ} = \boldsymbol{D}_{JE}^{-1}. \tag{72}$$

One could pass from the first integrand in each pair to the second one by taking the appropriate Legendre transform (see the discussion in Chapters 1 and 2).

To find the relation between the minimax and minimal variational principles we refer to the duality [**?**, **?**]. Any saddle function $f(x, y)$ of two variables x and y, corresponds through the Legendre transform x^* over the first variable x to the convex function $f_x^*(x^*, y)$ of the arguments x^*, y.

For example, the saddle function

$$f(x,y) = \frac{a}{2}x^2 - \frac{b}{2}y^2 \tag{73}$$

is conjugate in the variable x of the convex function

$$f_x^*(x^*, y) = \max_x [xx^* - f(x,y)] = \frac{1}{2a}(x^*)^2 + \frac{b}{2}y^2.$$

By using a similar idea, we can take the Legendre transform of the functional (60) over one of its variables (namely, over $\boldsymbol{E}''$) and obtain the minimal variational principle (68). Similarly, we can take the Legendre transform of the functional (63) over the variable $\boldsymbol{J}''$ and arrive at the minimum variational principle (65).

The relations between the four described variational problems are illustrated by the following scheme: The minimax variational principle

$$\min_{\boldsymbol{E}''} \max_{\boldsymbol{E}'} \left\{ U_{EE}(\boldsymbol{E}', \boldsymbol{E}'') \right\} \tag{74}$$

is transformed by the Legendre transform over the variable $\boldsymbol{E}''$ into the minimum variational principle

$$\min_{\boldsymbol{J}''} \min_{\boldsymbol{E}'} \left\{ U_{EJ}(\boldsymbol{E}', \boldsymbol{J}'') \right\}. \tag{75}$$

The next Legendre transform over the variable $\boldsymbol{E}'$ leads to the minimax variational problem

$$\min_{\boldsymbol{J}'} \max_{\boldsymbol{J}''} \left\{ -U_{JJ}(\boldsymbol{J}', \boldsymbol{J}'') \right\}, \tag{76}$$

which is equivalent to (62). The next transform over the variable $\boldsymbol{J}''$ leads to the maximization problem

$$\max_{\boldsymbol{J}'} \max_{\boldsymbol{E}''} \left\{ -U_{JE}(\boldsymbol{J}', \boldsymbol{E}'') \right\}, \tag{77}$$

which is equivalent to (65). If we take one more Legendre transform over the variable $\boldsymbol{J}'$, we arrive at a problem that coincides with the one with which we started.

The same method can be used to formulate the minimization problem for other problems described by equations with complex coefficients. For example, the equations of torsion oscillation of a bar made of viscoelastic materials coincide with (29)–(32) with some changes in the definitions of the fields and moduli [**?**]. The other important example of the complex moduli problem is given by viscoelasticity equations.