Optimal design: Problems with differential constraints

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1 Pointwise constraints: Optimal design

1.1 Stationarity conditions

Consider an optimal design problem. The statement of the problem requires the definition of the goal functional, differential constraints (the equations of the equilibrium of dynamics), the possible integral constraints, and the set $\mathcal{K}$ of controls. It may look as follows: Minimize a functional of the type

$$\Phi = \min_{k(x)\in\mathcal{K}} \left[\int_\Omega F(k,u)\,dx + \int_{\partial\Omega} F_\partial(k,u)\,ds \right] \tag{1}$$

where $u = u(x_1, \ldots, x_d)$ is a solution of a partial differential equation

$$Q(k, \nabla u, u) = 0 \ \text{ in } \Omega, \quad u = u_0 \ \text{ on } \partial\Omega \tag{2}$$

The approach with pointwise algebraic or differential constraints is similar those used in to one-dimensional variational problems. We construct an augmented functional I, adding to the functional the differential equation multiplied by a Lagrange multiplier $\lambda(x)$- the function of x, as follows

$$I = \min_{k(x)\in\mathcal{K}} \left[\min_{u:\ u=u_0 \text{ on } \partial\Omega} \left(\max_{\lambda} J(u, k, \lambda) \right) \right] \tag{3}$$

$$J = \int_\Omega [F(k, u) + \lambda\, Q(\nabla u, u)]\, dx + \int_{\partial\Omega} F_\partial(k, u_0)\, ds \tag{4}$$

We arrive at the variational problem with three variables, u, k, and λ. The stationarity condition with respect of μ is equation (5). The other two conditions are

$$S_u(F + \lambda Q) = 0, \quad S_\lambda(F + \lambda Q) = 0, \ \text{ in } \Omega \tag{5}$$

and corresponding boundary conditions.

Below we consider two simple minimization problems with differential constraints that express a thermal equilibrium. The equilibrium depends on the control (thermal sources or boundary temperature) that must be chosen to minimize a functional related to the temperature distribution. The differential constraint is the conductivity equation; it relates the temperature and control.

1.2 Design of boundary temperature

Consider the following problem: A bounded domain Ω is in thermal equilibrium. The temperature on its boundary $\theta(s)$ must be chosen to minimize the L_2 norm of deflection of the temperature T from a given target distribution $\rho(x)$.

Let us formulate the problem. The objective is

$$I = \min_T \frac{1}{2} \int_\Omega (T - \rho)^2 dx. \tag{6}$$

Temperature T is a solution to the boundary value problem of thermal equilibrium (the differential constraint)

$$\nabla^2 T = 0 \quad \text{ in } \Omega, \quad T = \theta \quad \text{ on } \partial\Omega. \tag{7}$$

Here, control θ enters the problem through the boundary condition of the differential constraint. The constraint connects control $\theta(s)$ with the variable $T(x)$, which is needed to compute the objective. The set of controls θ is an open set of all piece-wise differentiable functions.

Harmonic target The problem becomes trivial when target ρ is harmonic, $\nabla^2\rho = 0$ in Ω. In this case, we simply set $T = \rho$ everywhere in Ω and in particular at the boundary. The differential constraint is thus satisfied. The cost of the problem is zero, which mean that the global minimum is achieved.

Nonharmonic target: Stationarity We account for the first equation (7) as for the pointwise constraint. The Lagrange multiplier for the differential constraint (called also the adjoint variable) $\lambda(x)$ is a function of a point of the domain, because the constraint is enforced everywhere there. The augmented functional is

$$I_A = \int_\Omega \left(\frac{1}{2}(T-\rho)^2 + \lambda \nabla^2 T \right) dx. \tag{8}$$

It reaches stationarity at the optimal solution. To compute variation of I_A with respect to T, we twice integrate by parts the last term. The computation gives the following

$$\delta I_A = \int_\Omega \left(T - \rho + \nabla^2 \Lambda\right) \delta T \, dx + \oint_{\partial\Omega} \left[\left(\delta \frac{\partial T}{\partial n} \right) \lambda - \delta T \left(\frac{\partial}{\partial n} \lambda \right) \right] ds \tag{9}$$

The variation leads to the stationarity condition, which here has the form of the boundary value problem for λ,

$$\nabla^2 \lambda = \rho - T \quad \text{ in } \Omega, \tag{10}$$

and

$$\lambda = 0, \quad \frac{\partial}{\partial n} \lambda = 0 \quad \text{ on } \partial\Omega \tag{11}$$

Notice that the variations (see (7)) of the value of T and its normal derivative $\frac{\partial T}{\partial n}$ at the boundary $\partial\Omega$ are arbitrary because the control θ is not constrained, therefore the coefficients by these variations must be zero at the stationary solution.

Remark 1.1 The problem (10) for the dual variable λ has two boundary conditions, and the problem (7) for T has none! That is fine because these two problems are solved as a system of two second-order partial differential equations with two boundary conditions (11).

The primary problem is underdetermined because the control cannot be specified without the dual problem. The dual problem is overdetermined because it determines the control. The system of the pair of problem is well-posed.

Analysis To solve the system of necessary conditions, we first exclude T by taking Laplacian ∇^2 of the left- and right-hand side of the equation (10) and accounting for (7). Thus, we obtain a regular fourth-order boundary value problem for λ

$$\nabla^4 \lambda = \nabla^2 \rho \quad \text{ in } \Omega, \quad \lambda = 0, \quad \frac{\partial}{\partial n} \lambda = 0 \quad \text{ on } \partial\Omega. \tag{12}$$

that has a unique solution. After finding λ, we find T from (10). Then we compute the boundary values $\theta = T|_{\partial\Omega}$ and define the control.

Notice that if the target is harmonic, $\nabla^2 \rho = 0$, then (12) gives $\lambda = 0$ and (10) gives $T = \rho$, as expected.

1.3 Design of bulk sources

Consider again problem (6) of the best approximation of the target temperature. This time, consider control of bulk sources. Namely, assume that the heat sources $\mu = \mu(x)$ can be applied everywhere in the domain Ω but the boundary temperature is kept equal zero. Assume in addition, that the L_2 norm of the sources is bounded.

In this case, T is a solution to the boundary value problem (the differential constraint)

$$\nabla^2 T = \mu \quad \text{in } \Omega, \quad T = 0 \quad \text{on } \partial\Omega \tag{13}$$

and μ is bounded by an integral constraint

$$\frac{1}{2}\int_\Omega \mu^2 dx = A \tag{14}$$

but not pointwise. These constraints are accounted with Lagrange multipliers $\lambda(x)$ and γ, respectively. The extended functional depends on two functions T and μ,

$$\int_\Omega L(T,\mu)dx - \gamma A\|\Omega\|$$

where

$$L(T,\mu) = \frac{1}{2}(T-\theta)^2 + \lambda(\nabla^2 T - \mu) + \frac{1}{2}\gamma\mu^2 \tag{15}$$

The variations of L with respect to T and μ lead to stationary conditions. The stationarity with respect to T results in the boundary value problem for λ,

$$\nabla^2 \lambda = T - \theta \quad \text{in } \Omega, \quad \lambda = 0 \quad \text{on } \partial\Omega \tag{16}$$

The stationarity with respect to variation of μ leads to the pointwise condition

$$\lambda = -\gamma\mu$$

that allows to exclude μ from (13). and obtain the linear system

$$\begin{cases} \nabla^2 T - \frac{1}{\gamma}\lambda, \quad \nabla^2\lambda = T - \theta & \text{if } x \in \Omega \\ T = 0, \quad \lambda = 0 & \text{if } x \in \partial\Omega \end{cases} \tag{17}$$

and an integral constraint

$$\frac{1}{2}\int_\Omega \lambda^2 dx = \frac{A\|\Omega\|}{\gamma^2}.$$

that scales λ and, therefore, control μ. This system could be solved for $T(x)$, $\lambda(x)$ and the constant γ, which would completely define the solution.

Problem 1.1 Reduce the system to one fourth-order equation as in the previous problem. Derive boundary conditions. Using Green's function, obtain the integral representation of the solution through the target.

2 Optimal Conducting Structures

In this chapter we consider minimization of functionals that depend on the solution of the stationary conductivity problem. Energy minimization is one such problem. Other examples include minimization of the mean temperature within some area in the body, minimization of a norm of the difference between the desired and actual temperature, or maximization of the total current through a boundary component. Again, microstructures appear in these optimal designs. We describe an approach based on homogenization and demonstrate that the optimal composites are surprisingly simple: Laminates are the optimal structures for a large class of cost functionals.

We are turning toward a construction of minimizers for problem (73), (74). Here, we introduce the formal solution scheme to the problem and demonstrate that optimal designs correspond to the laminate structures. We describe a method suggested in [**?**]: The constrained extremal problem is reduced to a minimal unconstrained variational problem similar to the problem of energy minimization.

For definiteness, we consider a weakly continuous functional of the type

$$I(w) = \int_\Omega F(w) + \oint_{\partial\Omega} \Phi(w, w_n), \tag{18}$$

where F and Φ are differentiable functions. As an example, one can think of the problem of minimizing a temperature in a region inside the domain Ω.

2.1 Augmented Functional

Consider the minimization problem

$$I = \min_{\boldsymbol{\sigma}} \int_\Omega F(w) + \oint_{\partial\Omega} \Phi(w, w_n) \tag{19}$$

where w is the solution to system (73). Assume that the functional (18) is weakly continuous. Therefore, the solution to (18) exists if the set $\mathcal{U}$ of controls is G-closed. Here we find that all optimal structures are laminates.

In dealing with the constrained optimization problem we use Lagrange multipliers to take into account the differential constraints. We construct the augmented functional I_A. Namely, we add to I the two differential equations (73) multiplied by the scalar Lagrange multiplier $\lambda = \lambda(\boldsymbol{x})$ and vector Lagrange multiplier $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{x})$, respectively,

$$G = \lambda(\nabla \cdot \boldsymbol{j} - q) + \boldsymbol{\kappa} \cdot (\nabla w - \boldsymbol{\sigma}^{-1}\boldsymbol{j}).$$

This way we obtain the augmented functional

$$I_A = \min_{\boldsymbol{\sigma}} \min_{w,\boldsymbol{j}} \max_{\lambda,\boldsymbol{\kappa}} \int_\Omega (F + G) + \oint_{\partial\Omega} \Phi(w, w_n). \tag{20}$$

The boundary conditions (73) are assumed. The value of the augmented functional I_A is equal to the value of I; see, for example, [**?**].

The Adjoint Problem To find the Lagrange multipliers λ and $\boldsymbol{\kappa}$ we calculate the variation of the augmented functional (20) caused by the variation δw of w and the variation $\delta \boldsymbol{j}$ of $\boldsymbol{j}$. The multipliers satisfy the stationarity condition $\delta I_A = 0$, where

$$\begin{aligned} \delta I_A \; = & \int_\Omega \left(\frac{\partial F}{\partial w} \delta w + \lambda (\nabla \cdot \delta \boldsymbol{j}) + \boldsymbol{\kappa} \cdot (\nabla \delta w - \boldsymbol{\sigma}^{-1} \delta \boldsymbol{j}) \right) \\ & + \oint_{\partial \Omega} \left(\frac{\partial \Phi}{\partial w} \delta w + \frac{\partial \Phi}{\partial j_n} \delta j_n \right). \end{aligned}$$

We apply the standard variational technique, transforming the terms on the right-hand side using Green's theorem:

$$\int_\Omega (a \nabla \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \nabla a) = \oint_{\partial \Omega} a b_n \tag{21}$$

where $a, \boldsymbol{b}$ are differentiable scalar and vector functions, respectively; $b_n = \boldsymbol{b} \cdot \boldsymbol{n}$ is the normal component of $\boldsymbol{b}$, and Ω is a domain with a smooth boundary $\partial \Omega$.

We apply this theorem to the two terms on the right-hand side of the expression for δI_A:

$$\begin{aligned} \int_\Omega \lambda \nabla \cdot \delta \boldsymbol{j} \; &= - \int_\Omega \delta \boldsymbol{j} \cdot \nabla \lambda + \oint_{\partial \Omega} \lambda \delta j_n, \\ \int_\Omega \boldsymbol{\kappa} \cdot \nabla \delta w \; &= - \int_\Omega \delta w \nabla \cdot \boldsymbol{\kappa} + \oint_{\partial \Omega} \kappa_n \delta w. \end{aligned}$$

This brings δI_A to the form

$$\delta I_A = \int_\Omega (\mathcal{A} \delta w + \mathcal{B} \cdot \delta \boldsymbol{j}) + \oint_{\partial \Omega} (\mathcal{C} \delta w + \mathcal{D} \delta j_n), \tag{22}$$

where

$$\begin{array}{ll} \mathcal{A} = -\nabla \cdot (\boldsymbol{\kappa}) + \frac{\partial F}{\partial w}, & \mathcal{B} = -\boldsymbol{\sigma}^{-1} \boldsymbol{\kappa} - \nabla \lambda, \\ \mathcal{C} = \frac{\partial \Phi}{\partial w} + \kappa_n, & \mathcal{D} = \frac{\partial \Phi}{\partial j_n} + \lambda. \end{array}$$

The stationarity $\delta I_A = 0$ of I_A with respect to the variations δw and $\delta \boldsymbol{j}$, together with the boundary conditions (73), implies

$$\begin{array}{ll} \mathcal{A} = 0 \;\; \text{in } \Omega, & \mathcal{B} = 0 \;\; \text{in } \Omega, \\ \mathcal{C} = 0 \;\; \text{on } \partial \Omega_1, & \mathcal{D} = 0 \;\; \text{on } \partial \Omega_2. \end{array}$$

The pair λ, $\boldsymbol{\kappa}$ satisfies the adjoint problem (compare with (73))

$$\left. \begin{array}{r} \nabla \cdot \boldsymbol{\kappa} = -\frac{\partial F}{\partial w} \\ \sigma(\chi) \nabla \lambda = -\boldsymbol{\kappa} \end{array} \right\} \text{ in } \Omega, \qquad \begin{array}{l} \lambda = -\frac{\partial \Phi}{\partial j_n} \;\; \text{on } \partial \Omega_1, \\ \kappa_n = -\frac{\partial \Phi}{\partial w} \;\; \text{on } \partial \Omega_2. \end{array} \tag{23}$$

To obtain these boundary conditions we notice that $\delta w|_{\partial \Omega_1} = 0$ and $\delta j_n|_{\partial \Omega_2} = 0$.

System (23) allows us to determine λ for given $\boldsymbol{\sigma}$ and w. The multipliers λ and $\boldsymbol{\kappa}$ (23) are similar to the variables w and $-\boldsymbol{j}$ (73), respectively, because they correspond to the same inhomogeneous layout $\boldsymbol{\sigma}$ but to a different right-hand-side term and boundary conditions.

Self-Adjoint Problem An important special case of problem (20) is the coincidence of λ and w. This happens if

$$\frac{\partial F}{\partial w} = q, \quad \frac{\partial \Phi}{\partial j_n} = -\rho_1, \quad \frac{\partial \Phi}{\partial w} = \rho_2. \tag{24}$$

The solutions to these problems also coincide:

$$\lambda = w, \quad \boldsymbol{j} = -\boldsymbol{\kappa}. \tag{25}$$

In this case we call the problem *self-adjoint*. The self-adjoint problem corresponds to the special functional (19):

$$F(w) = qw, \quad \Phi = \begin{cases} -\rho_1 w & \text{on } \ \partial\Omega_1, \\ \rho_2 j_n & \text{on } \ \partial\Omega_2. \end{cases} \tag{26}$$

The reader can check that this functional represents the total energy stored in the body.

Also, we deal with a self-adjoint problem when the problem for a Lagrange multiplier λ differs only by sign from the variable w:

$$\lambda = -w, \quad \boldsymbol{j} = \boldsymbol{\kappa}. \tag{27}$$

This happens when the negative of the value of the work is minimized, or, equivalently, when the work is maximized; in this case both integrands F and Φ are negatives of those given by (26).

The self-adjoint problems minimize or maximize the energy of a structure; they were discussed in Chapter 4.

2.2 The Local Problem

Equations (73) and (23) determine two of three unknown functions w, λ, and $\boldsymbol{\sigma}$. The remaining problem is to find an additional relation among $\boldsymbol{\sigma}$, w, and λ.

Remark 2.1 In the exposition, we will associate the conductivity problem with the potential w assuming that $\boldsymbol{j}$ is computed from (73). Similarly, we associate the adjoint problem with the potential λ assuming that $\boldsymbol{\kappa}$ is computed from (23).

We use the homogenization approach. Let us average the augmented functional. We bring problem (19) to a symmetric form by integration by parts of the term that contains the control $\boldsymbol{\sigma}$:

$$I_A = \min_{\boldsymbol{\sigma}} \min_{w} \max_{\lambda} \int_\Omega (F(w) - q\lambda - \nabla\lambda \cdot \boldsymbol{\sigma}\nabla w) + \text{boundary terms}, \tag{28}$$

and we apply the averaging operator (**??**) $\langle\ \rangle$ to it.

In performing the averaging we replace (28) by the homogenized problem

$$I_A^\varepsilon = \min_{\sigma} \min_{\langle w\rangle} \max_{\langle\lambda\rangle} P,$$

where

$$P = \int_\Omega (\langle F(w)\rangle - \langle q\lambda\rangle - \langle \nabla\lambda\sigma \cdot \nabla w\rangle) + \text{boundary terms}. \tag{29}$$

The cost of the homogenized problem is arbitrarily close to the cost of the original problem:

$$I_A^\varepsilon \to I_A \quad \text{when } \varepsilon \to 0.$$

It is easy to compute the first two terms of the integrand in the volume integral on the right-hand side of (29), supposing that the functions $F(w)$ and q are continuous:

$$\langle F(w)\rangle = F(\langle w\rangle) + o(\varepsilon) \quad \text{and} \quad \langle q \cdot \lambda\rangle = q \cdot \langle \lambda\rangle + o(\varepsilon). \tag{30}$$

Calculation of the remaining term,

$$\langle \nabla\lambda\sigma \cdot \nabla w\rangle) = K(\nabla\lambda, \nabla w),$$

where

$$K(\nabla\lambda, \nabla w) = \frac{1}{\|\omega\|} \min_{\boldsymbol{\sigma}(\boldsymbol{x})} \min_{w} \max_{\lambda} \int_\omega \nabla(-\lambda)(\boldsymbol{\sigma}(\boldsymbol{x}))\nabla w, \tag{31}$$

requires homogenization. Term K should be expressed as a function of the averaged fields

$$\boldsymbol{p} = \langle \nabla w\rangle \quad \text{and} \quad \boldsymbol{q} = \langle \nabla\lambda\rangle \tag{32}$$

as $K = K(\boldsymbol{p}, \boldsymbol{q})$.

This problem is called the *local problem.* It asks for an optimal layout $\boldsymbol{\sigma}$ in the element of periodicity ω (considered a neighborhood of a point of the body Ω in the large scale).

Fixed Volume Fractions First, we solve an auxiliary problem. Consider the functional K (31) and assume that the volume fraction m of the first material in the structure is also given. The cost of (31) can be expressed through the tensor of effective properties $\boldsymbol{\sigma}_*$. It takes the form

$$J(\boldsymbol{p}, \boldsymbol{q}, m) = \min_{\boldsymbol{\sigma}_* \in G_m U} \langle \boldsymbol{p} \cdot \boldsymbol{\sigma}_* \boldsymbol{q}\rangle \tag{33}$$

and asks for the best structure of a composite with the fixed volume fractions of components. The volume fraction m will be determined later to minimize the cost of (31). Note that

$$K(\boldsymbol{p}, \boldsymbol{q}) = \min_{m \in [0,1]} J(\boldsymbol{p}, \boldsymbol{q}, m). \tag{34}$$

We calculate $J(\boldsymbol{p}, \boldsymbol{q}, m)$, referring only to the Wiener bounds; the eigenvalues λ_i of any effective tensor $\boldsymbol{\sigma}$ are bounded as

$$\sigma_h \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \sigma_a. \tag{35}$$

Note that the bounds of the G_m-closure restrict eigenvalues of the effective tensor but not its orientation. Although these bounds are uncoupled and therefore not complete, they provide enough information to solve the optimization problem.

Let us analyze the bilinear form J. It depends only on magnitudes $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ and the angle 2θ between them:

$$J = \min_{\boldsymbol{\sigma} \in G_m U} \boldsymbol{p} \cdot \boldsymbol{\sigma} \boldsymbol{q} = J(|\boldsymbol{p}|,\ |\boldsymbol{q}|, \theta), \tag{36}$$

where

$$\theta = \frac{1}{2} \arccos \frac{|\boldsymbol{p} \cdot \boldsymbol{q}|}{|\boldsymbol{p}|\, |\boldsymbol{q}|}. \tag{37}$$

Next, J is proportional to the magnitudes $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ of both external fields $\boldsymbol{p}$ and $\boldsymbol{q}$; it can be rewritten in the form

$$J(\boldsymbol{p}, \boldsymbol{q}, m) = |\boldsymbol{p}||\boldsymbol{q}| \min_{\boldsymbol{\sigma}_* \in G_m U} \boldsymbol{i}_p \cdot \boldsymbol{\sigma}_* \boldsymbol{i}_q, \tag{38}$$

where the unit vectors $\boldsymbol{i}_p$ and $\boldsymbol{i}_q$ show the directions of $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{i}_p = \frac{1}{|\boldsymbol{p}|}\boldsymbol{p}, \quad \boldsymbol{i}_q = \frac{1}{|\boldsymbol{q}|}\boldsymbol{q}. \tag{39}$$

Thus (38) is reduced to the minimization of a nondiagonal $p\,q$-component of the tensor $\boldsymbol{\sigma} \in G_m U$ in the nonorthogonal coordinate system of axes $\boldsymbol{i}_p$ and $\boldsymbol{i}_q$.

Diagonalization To find the optimal orientation of the tensor $\boldsymbol{\sigma}$ we introduce a pair of vectors

$$\boldsymbol{a} = \boldsymbol{i}_p + \boldsymbol{i}_q, \quad \boldsymbol{b} = \boldsymbol{i}_p - \boldsymbol{i}_q. \tag{40}$$

Due to normalization (39), these fields are orthogonal: $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

The magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ depend only on θ (see (37)). They equal $|\boldsymbol{a}| = 2\cos\theta, \quad |\boldsymbol{b}| = 2\sin\theta$ (see **??**).

In the introduced notation, the bilinear form J takes the form of the difference of two quadratic functions,

$$J = \min_{\boldsymbol{\sigma} \in G_m U} |\boldsymbol{p}||\boldsymbol{q}|[\boldsymbol{a} \cdot \boldsymbol{\sigma} \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{\sigma} \boldsymbol{b}]. \tag{41}$$

Optimal Effective Tensor The minimization of J over $\boldsymbol{\sigma} \in G_m U$ requires the optimal choice of the eigenvectors and eigenvalues of $\boldsymbol{\sigma}$. To minimize J, we direct the eigenvectors of $\boldsymbol{\sigma}$ as follows. The eigenvector of $\boldsymbol{\sigma}$ that corresponds to the minimal eigenvalue $\lambda_{\min}$ is oriented parallel to $\boldsymbol{a}$, and the eigenvector of $\boldsymbol{\sigma}$ that corresponds to the maximal eigenvalue $\lambda_{\max}$ is oriented parallel to $\boldsymbol{b}$:

$$J \geq |\boldsymbol{p}||\boldsymbol{q}|(\lambda_{\min} \boldsymbol{a}^2 - \lambda_{\max} \boldsymbol{b}^2). \tag{42}$$

Next, J decreases if $\lambda_{\min}$ coincides with its lower bound, and $\lambda_{\max}$ coincides with its upper bound; the bounds are given by (35). We have

$$J \geq |\boldsymbol{p}||\boldsymbol{q}|(\sigma_h \boldsymbol{a}^2 - \sigma_a \boldsymbol{b}^2). \tag{43}$$

Remark 2.2 The value of the intermediate eigenvalue is irrelevant. This feature recalls the definition of the weak G-closure (Chapter 3). The weak G-closure is adequate for this problem because it deals with an arbitrary pair of the field and the current.

Formula (43) displays the basic qualitative property of an optimal composite:

> The optimal effective tensor $\boldsymbol{\sigma}_*$ possesses maximal difference between weighted maximal and minimal eigenvalues. The optimal structures are extremely anisotropic.

Substituting the values of the original fields $\boldsymbol{a}$ and $\boldsymbol{b}$ into (43), we find that J is bounded from below as follows:

$$J \geq 2[(\sigma_a + \sigma_h)(|\boldsymbol{p}||\boldsymbol{q}| - (\sigma_a - \sigma_h)\boldsymbol{p} \cdot \boldsymbol{q}). \tag{44}$$

Inequality (44) is valid for all composites independent of their structure.

The fields $\boldsymbol{p}$ and $\boldsymbol{q}$ lie in the plane of maximal and minimal eigenvalues of $\boldsymbol{\sigma}_*$. The direction of $\lambda_{\min}$ bisects the angle between the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. Qualitatively speaking, $\boldsymbol{p}$ shows the attainable direction of currents, $\boldsymbol{q}$ shows the desired direction of them. The direction of maximal conductivity in the optimal structure bisects this angle. This rule provides a compromise between the availability and the desire.

Remark 2.3 Note that the self-adjoint problem (25), where $\boldsymbol{p} = \boldsymbol{q}$, corresponds to $\boldsymbol{a} = 0$ and the problem where $\boldsymbol{p} \equiv -\boldsymbol{q}$ (27) corresponds to $\boldsymbol{b} = 0$.

Optimal Structures An appropriately oriented laminate provides the minimal value of J. Indeed, the laminates have simultaneously the maximal conductivity σ_a in a direction(s) (along the layers) and the minimal conductivity σ_h in the perpendicular direction (across the layers).

The optimal laminates are oriented so that the normal $\mathbf{n}$ coincides with the direction of the field $\boldsymbol{a}$, and the tangent $\mathbf{t}$ coincides with the direction of $\boldsymbol{b}$. The cost $J(\boldsymbol{\sigma}_{\mathrm{lam}})$ of the local problem for laminate structure $\boldsymbol{\sigma}_{\mathrm{lam}}$ coincides with the bound (44).

2.3 Solution in the Large Scale

The solution to the auxiliary local problem allows us to compute K from (34). We denote $m_1 = m,\ m_2 = 1-m$; assume that $\sigma_2 > \sigma_1$; and calculate an optimal value of the volume fractions of materials in the laminates. We have

$$\begin{aligned}\frac{K}{|\boldsymbol{p}|\,|\boldsymbol{q}|} &= \min_{m\in[0,1]} \left(\lambda_{\min}(m)\boldsymbol{a}^2 - \lambda_{\max}(m)\boldsymbol{b}^2\right) \\ &= \min_{m\in[0,1]} \left(\frac{\sigma_1\sigma_2}{m\sigma_2 + (1-m)\sigma_1}\boldsymbol{a}^2 + (m\sigma_1 + (1-m)\sigma_2)\boldsymbol{b}^2\right). \end{aligned} \tag{45}$$

The optimal value $m^{\rm opt}$ of m depends only on the ratio between $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$,

$$\frac{|\boldsymbol{a}|}{|\boldsymbol{b}|} = \cot\theta, \tag{46}$$

and is equal to

$$m^{\rm opt} = \begin{cases} 0 & \text{if } \ \cot\theta \le \sqrt{\frac{\sigma_1}{\sigma_2}}, \\ \frac{\sqrt{\sigma_1\sigma_2}}{\sigma_2-\sigma_1}\left(\cot\theta - \sqrt{\frac{\sigma_1}{\sigma_2}}\right) & \text{if } \ \sqrt{\frac{\sigma_1}{\sigma_2}} \le \cot\theta \le \sqrt{\frac{\sigma_2}{\sigma_1}}, \\ 1 & \text{if } \ \cot\theta \ge \sqrt{\frac{\sigma_2}{\sigma_1}}. \end{cases} \tag{47}$$

Equation (47) says that the optimal concentration of materials in the laminates depends only on the angle θ.

We find the optimal value of the functional $K = J(m^{\rm opt})$:

$$\frac{K}{|\boldsymbol{p}|\,|\boldsymbol{q}|} = \begin{cases} \sigma_2\cos 2\theta & \text{if } \ \cot\theta \le \sqrt{\frac{\sigma_1}{\sigma_2}}, \\ (\sigma_1+\sigma_2)\sin^2\theta & \text{if } \ \sqrt{\frac{\sigma_1}{\sigma_2}} \le \cot\theta \le \sqrt{\frac{\sigma_2}{\sigma_1}}, \\ \sigma_1\cos 2\theta & \text{if } \ \cot\theta \ge \sqrt{\frac{\sigma_2}{\sigma_1}}. \end{cases} \tag{48}$$

To complete the solution, it remains to pass to the original notation $\nabla\langle w\rangle = \boldsymbol{p}$ and $\nabla\langle\lambda\rangle = \boldsymbol{q}$, substitute the value of the local problem into the functional (20), and find the Euler–Lagrange equations of the problem:

$$I_A = \min_{\langle w\rangle}\max_{\langle\lambda\rangle}\int_{\mathcal{O}}[F(\langle w\rangle) + \langle\lambda\rangle q + K]. \tag{49}$$

Note that the equations for $\langle w\rangle$ and $\langle\lambda\rangle$ are coupled because the optimal properties depend on both of them.

Numerical Procedure Practically, we have used a different procedure for the numerical solution; see [**?**]. The iterative method has been organized as follows:

1. Given a layout of $\boldsymbol{\sigma}$, we compute the solution w of problem (73) and the solution λ of the adjoint problem (23).
2. The optimal layouts $m(\boldsymbol{x})$ and $\theta(\boldsymbol{x})$ is found from (46). Then we return to the first step.

3 Reducing to a Minimum Variational Problem

Duality In this section, we reformulate the local minimax problem (41) as a minimal variational problem. This way we reduce the problem to the type discussed in Chapter 4. The relaxation is obtained by the convexification. The transformation is done by the Legendre transform.

The Legendre transform replaces the saddle Lagrangian (41) with a convex Lagrangian. Recall (chapter 1) that the Legendre transform or Young–Fenchel transform $f^*(x^*)$ (**??**) of a concave function $f(x)$ is given by (**??**),

$$f^*(x^*) = \min_x [x\, x^* - f(x)], \tag{50}$$

where $f^*(x^*)$ is the conjugate to $f(x)$.

Consider the Legendre transform of the function

$$f(b) = \frac{\lambda_1}{2} a^2 - \frac{\lambda_2}{2} b^2, \tag{51}$$

where a is a parameter. We compute

$$f_b^*(a, b^*) = \min_b [b\, b^* - f(a,b)] = \frac{\lambda_1}{2}(a)^2 + \frac{1}{2\lambda_2} {b^*}^2, \quad b^* = \lambda_2 b. \tag{52}$$

Notice that $f_b^*(a, b^*)$ is a convex function of a and b^*

Consider the normalized local minimax problem (see (41))

$$J = \min_{\boldsymbol{\sigma}_* \in G_m U} \left\{ \frac{1}{2} \min_{\boldsymbol{a}} \max_{\boldsymbol{b}} \left\{ \langle \boldsymbol{a} \rangle \cdot \boldsymbol{\sigma}_* \langle \boldsymbol{a} \rangle - \langle \boldsymbol{b} \rangle \cdot \boldsymbol{\sigma}_* \langle \boldsymbol{b} \rangle \right\} \right\}, \tag{53}$$

where we put $|\boldsymbol{p}| = |\boldsymbol{q}| = 1$,

To reduce (53) to a minimum problem, we perform the Legendre transform on the variable $\boldsymbol{b}$. We replace the problem of maximization of conductivity in the direction $\boldsymbol{b}$ with the problem of the minimization of the resistance in this direction. This transform does not include optimization; we simply change the variable $\boldsymbol{b}$ to its conjugate variable $\boldsymbol{b}^*$.

The dual to $\boldsymbol{b}$ variable $\boldsymbol{b}^* = \boldsymbol{j}$ is (see (52)):

$$\boldsymbol{j} = \boldsymbol{\sigma} \boldsymbol{b}. \tag{54}$$

If we substitute (54) into (53), the latter becomes the following minimum problem:

$$\begin{aligned} R_*(\boldsymbol{a}, \boldsymbol{j}) &= \min_{\boldsymbol{\sigma}} R(\boldsymbol{a}, \boldsymbol{j}, \boldsymbol{\sigma}), \\ R(\boldsymbol{a}, \boldsymbol{j}, \boldsymbol{\sigma}) &= \left\langle \left(-\boldsymbol{b} \cdot \boldsymbol{j} + \frac{1}{2} \boldsymbol{a} \cdot \boldsymbol{\sigma} \boldsymbol{a} + \frac{1}{2} \boldsymbol{j} \cdot \boldsymbol{\sigma}^{-1} \boldsymbol{j} \right) \right\rangle. \end{aligned} \tag{55}$$

Recall that the eigenvectors of $\boldsymbol{\sigma}$ are oriented along $\boldsymbol{a}$ and $\boldsymbol{b}$. Therefore, the vectors $\boldsymbol{j}$ and $\boldsymbol{b}$ are parallel, and $\boldsymbol{j}$ and $\boldsymbol{a}$ are orthogonal.

The reformulated local problem (55) asks for a medium that stores *the minimal sum of the energy and the complementary energy* caused by two orthogonal external fields.

Minimal Variational Problem The function R is a solution to a variational problem for unknown fields and layout, similar to the variational problem of Chapter 4.

This local variational problem can be rewritten as:

$$R(\boldsymbol{a}, \boldsymbol{b}, m) = \min_{\chi} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \min_{\boldsymbol{\beta} \in \mathcal{B}} J_R(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi),$$

where χ is subject to the constraint $\langle \chi \rangle = m$, and

$$J_R(\boldsymbol{\alpha}, \boldsymbol{\beta}, \chi) = \langle -\boldsymbol{\alpha} \cdot \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}(\chi) \boldsymbol{\alpha} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{\sigma}(\chi)^{-1} \boldsymbol{\beta} \rangle, \tag{56}$$

$\mathcal{A}$ is the set of periodic gradient fields with mean value $\boldsymbol{a}$,

$$\mathcal{A} = \{ \boldsymbol{\alpha} : \ \nabla \times \boldsymbol{\alpha} = 0, \ \langle \boldsymbol{\alpha} \rangle = \boldsymbol{a}, \ \boldsymbol{\alpha}(\boldsymbol{x}) \text{ is } \Omega - \text{periodic} \}, \tag{57}$$

and $\mathcal{B}$ is the set of divergencefree periodic vectors with the mean value $\boldsymbol{b}$,

$$\mathcal{B} = \{ \boldsymbol{\beta} : \ \nabla \cdot \boldsymbol{\beta} = 0, \ \langle \boldsymbol{\beta} \rangle = \boldsymbol{j}, \ \boldsymbol{\beta}(\boldsymbol{x}) \text{ is } \Omega - \text{periodic} \}. \tag{58}$$

Note that variational problem (56) does not contain differential constraints. As with the problem in Chapter 4, this problem can be analyzed by classical variational methods.

Remark 3.1 The minimal form J_R of the Lagrangian is symmetric in the sense that the field $\boldsymbol{a}$ and the current $\boldsymbol{j}$ enter into the problem in the same way. One expects this property because the original conductivity problem could be formulated in two equivalent ways: By using a field potential or a current potential; the result is unrelatedto this choice. Both self-adjoint cases correspond to either $\boldsymbol{a} = 0$ or $\boldsymbol{j} = 0$. The problem is reduced to minimization of the energy in one of the dual forms, as one would expect.

The Transferred Problem as a Nonconvex Variational Problem Problem (56) depends on χ and it needs a relaxation. The problem can be relaxed by convexification. As in the problem in Chapter 4, we can exclude the control χ by interchanging the minimal operations:

$$R(\boldsymbol{a}, \boldsymbol{b}, m) = \min_{\sigma(\chi)} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \min_{\boldsymbol{\beta} \in \mathcal{B}} J_R = \min_{\boldsymbol{\alpha} \in \mathcal{A}} \min_{\boldsymbol{\beta} \in \mathcal{B}} \min_{\sigma(\chi)} J_R. \tag{59}$$

The inner minimum $\min_{\sigma(\chi)} J_R$ can be easily computed, because χ takes only two values: zero and one. The problem becomes

$$R(\boldsymbol{a}, \boldsymbol{b}, m) = \min_{\boldsymbol{a} \in \mathcal{A}} \min_{\boldsymbol{j} \in \mathcal{B}} J_{RR}, \tag{60}$$

where

$$J_{RR}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left\langle -\boldsymbol{\alpha} \cdot \boldsymbol{\beta} + \frac{1}{2} \min \left\{ \sigma_1 |\boldsymbol{\alpha}|^2 + \frac{1}{\sigma_1} |\boldsymbol{\beta}|^2, \ \sigma_2 |\boldsymbol{\alpha}|^2 + \frac{1}{\sigma_2} |\boldsymbol{\beta}|^2 \right\} \right\rangle. \tag{61}$$

Notice that $J_{RR}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a nonconvex function of its arguments.

To relax problem (56) we again perform the convexification of the Lagrangian $J_{RR}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. It is a two-well Lagrangian, and therefore its convex envelope is supported by two points that belong to different wells. Following the calculation in Chapter 4, we find these points and show that the convex envelope is equal to

$$\mathcal{C}J_{RR}(\boldsymbol{a}, \boldsymbol{j}) = -\boldsymbol{a} \cdot \boldsymbol{j} + \frac{1}{2} \min_{m \in [0,1]} \left\{ \sigma_h(m)\ \boldsymbol{a}^2 + \frac{1}{\sigma_a(m)}\ \boldsymbol{j}^2 \right\}, \tag{62}$$

where $(\cdot)_h$ and $(\cdot)_a$ are again the harmonic and the arithmetic means. To obtain the term $\frac{1}{\sigma_a(m)}$, we use the identity $\left(\frac{1}{\sigma}\right)_h = \frac{1}{\sigma_a}$.

To compute the first term $\langle \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \rangle = \boldsymbol{a} \cdot \boldsymbol{j}$ on the right-hand side, we use the property of divergencefree and curlfree fields called *compensated compactness* (see Chapter 8 and [**?**, **?**]).

The convex envelope of the Lagrangian (61) is again attainable, this time due to the orthogonality of $\boldsymbol{a}$ and $\boldsymbol{j}$. Indeed, the laminates have conductivity σ_h and σ_a in orthogonal directions. The structure can be oriented so that the axis σ_h is directed along $\boldsymbol{a}$ and the axis σ_a along $\boldsymbol{j}$.

The Legendre transform is an involution: After the convexification is performed, we perform the transform with respect to $\boldsymbol{j}$ to bring the problem back to the form (41).

Remark 3.2 Another approach to these problems was developed in [**?**, **?**]. Instead of transforming the functional, the author developed a method for finding an saddle-type envelope of the Lagrangian of a minimax problem. Both approaches lead to similar results, as expected.

Summary of the Method Let us outline the basic steps of the suggested method of relaxation. We assume that the problem is described by self-adjoint elliptic equations; the shape of the domain, boundary conditions, and external loadings (right-hand-side terms) are fixed; the minimizing functional is weakly lower semicontinuous. To relax the problem, we follow this procedure:

1. Formulate the local problem as a min-min-max problem for bilinear form of the field and gradient of the Lagrange multiplier; the coefficients of the form are the effective properties of the composite.

2. Normalize the bilinear form and transform it to the diagonal form by introducing new potentials. A transformed problem asks for minimization of the difference of energies caused by two orthogonal fields by the layout of the materials.

3. Use the Legendre transform to reduce the problem to a minimal variational problem, ending up with a problem of the minimization of the sum of an energy and a complementary energy.

4. Use convexification to bound the minimized nonconvex functional from below and find a minimizing sequence (i.e., the optimal microstructure) that bounds it from above.

5. Return to the original notation performing the Legendre transform of the convexified problem.

4 Examples

Example 4.1 Consider a thin circular cylindrical shell of height h and radius r made of two conducting materials σ_1 and σ_2. Suppose that its upper and lower faces are kept by different potentials. Let us find a layout of materials that maximizes the circumferential component of the current; this component is zero for isotropic materials.

Introduce the rectangular coordinates z, Θ on the surface. The surface Ω is the rectangle

$$0 \le z \le h, \quad 0 \le \Theta \le 2\pi r.$$

The potential $w(z, \Theta)$ is the solution to the problem

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{ in } \Omega, \tag{63}$$

with the boundary conditions

$$w(0,\Theta) = 0,\ w(h,\Theta) = U,\ w(z,0) = w(z,\pi),\ \frac{\partial w(z,0)}{\partial \Theta} = \frac{\partial w(z,\pi)}{\partial \Theta}.$$

The maximizing functional is the circumferential component of the current; it is written as

$$I = \int_\Omega \boldsymbol{i}_\Theta \cdot \sigma \nabla w, \tag{64}$$

where $\boldsymbol{i}_\Theta$ is a unit vector that points in the circumferential direction.

Applying the preceding analysis, we find that the Lagrange multiplier λ satisfies the problem (see (23))

$$\nabla \cdot \sigma(\nabla \lambda + \boldsymbol{i}_\Theta) = 0 \quad \text{ in } \Omega,$$

with boundary conditions

$$\begin{array}{ll} \lambda(0,\Theta) = 0, & \lambda(h,\Theta) = 0, \\ \lambda(z,0) = \lambda(z,\pi), & \frac{\partial \lambda(z,0)}{\partial \Theta} = \frac{\partial \lambda(z,\pi)}{\partial \Theta}. \end{array} \tag{65}$$

The solution to the averaged problem is easily found. We observe that the constant tensor $\boldsymbol{\sigma}_*$ and constant fields

$$\langle \nabla \lambda \rangle = -\boldsymbol{i}_\Theta \quad \text{and} \quad \langle \nabla w \rangle = \frac{U}{h} \boldsymbol{i}_z, \quad \boldsymbol{\sigma}_* = \text{constant}(\boldsymbol{x}), \tag{66}$$

satisfy the equations and the boundary conditions due to the symmetry of the domain and the special boundary conditions. The same composite is used at each point of the domain.

The angle θ that bisects the direction of the fields $\langle \nabla\lambda \rangle$ and $\langle \nabla w \rangle$ is equal to $\frac{\pi}{4}$. The optimal volume fraction m_{opt} of the first material (see (48)) is

$$m_{\text{opt}} = \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}}. \tag{67}$$

The cost K of the local problem (see (48)) is equal to the maximal current across the acting field. This cost and the functional I are

$$K = \frac{U}{h}(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2, \quad I = \int_\Omega K = 2\pi r(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2 U. \tag{68}$$

Example 4.2 The next problem is more advanced. It deals with an inhomogeneous layout of optimal laminates. The problem has been formulated and solved in [**?**]; an exposition of the solution can be found in [**?**].

Consider a circular cylinder $(0 < r < 1,\ 0 < z < h)$. Suppose that the constant heat flux $-\boldsymbol{j} \cdot \boldsymbol{n}$ is applied to the upper face $(z = h)$. The lateral surface $(r = 1)$ of the cylinder is heat insulated, and the lower face $(z = 0)$ is kept at zero temperature.

The cylinder is filled with two materials with heat conductivities σ_1 and σ_2. The steady state is described by the boundary value problem

$$\begin{array}{rll} \boldsymbol{j} &= \sigma(r,z)\nabla T, \quad \nabla \cdot \boldsymbol{j} = 0, & \text{inside the cylinder,} \\ \boldsymbol{i}_z \cdot \boldsymbol{j} &= 1, & \text{on the upper face,} \\ T &= 0, & \text{on the lower face,} \\ \boldsymbol{i}_r \cdot \boldsymbol{j} &= 0, & \text{on the lateral surface.} \end{array} \tag{69}$$

Here T is the temperature; $\boldsymbol{\sigma}(r,z) = \chi(r,z)\sigma_1 + (1 - \chi_1(r,z))\sigma_2$; $\chi(r,z)$ is the characteristic function; $\boldsymbol{i}_r$ and $\boldsymbol{i}_z$ are the unit vectors directed along the axes of the cylindrical coordinate system.

It is required to lay out the materials in the domain to minimize the functional over the lower face

$$I = \int_0^1 \rho(r)\boldsymbol{i}_z \cdot \boldsymbol{j}|_{z=0}\, r\, dr.$$

Here $\rho(r)$ is a weight function. Notice that I is the boundary integral, and the problem asks for the optimization of boundary currents caused by fixed boundary potentials.

In particular, if the weight function $\rho(r)$ is

$$\rho(r) = \begin{cases} 1 & \text{if} \ \ 0 < r < r_0, \\ 0 & \text{if} \ \ r_0 < 1, \end{cases} \tag{70}$$

then the problem is transformed to maximization of the heat flux through a circular "window" of radius r_0 on the lower surface of the cylinder.

Assuming that the set of admissible controls contains the initial materials and the composites assembled from them, we relax the problem. The augmented functional of the problem has the following form:

$$J = \int_0^1 \rho(r)\boldsymbol{i}_z \cdot \boldsymbol{j}|_{z=0} r\,dr + 2\pi \int_0^1 \int_0^h \lambda \nabla \cdot \boldsymbol{\sigma}_* \nabla T\, r\,dr\,dz. \tag{71}$$

Varying (71) with respect to T and $\boldsymbol{j}$ and taking into consideration the boundary conditions (69), we obtain a boundary value problem for the conjugate variable λ:

$$\begin{aligned} \boldsymbol{\kappa} = \boldsymbol{\sigma}_* \nabla\lambda, \quad \nabla \cdot \boldsymbol{\kappa} &= 0 && \text{inside the cylinder,} \\ \boldsymbol{\kappa} \cdot \boldsymbol{i}_r &= 0 && \text{on the lateral surface,} \\ \boldsymbol{\kappa} \cdot \boldsymbol{i}_z &= 0 && \text{on the upper face,} \\ \lambda &= \rho(r) && \text{on the lower face.} \end{aligned} \tag{72}$$

Problem (72) describes the "temperature" λ generated by the prescribed boundary values of λ on the lower face $z = 0$ if the upper face and lateral surface of the cylinder are heat insulated.

According to our analysis, an optimal layout of materials is characterized by a zone of material of low conductivity if the directions of gradients ∇T and $\nabla\lambda$ are close to each other, by a zone of material of high conductivity if the angle between directions of gradients ∇T and $\nabla\lambda$ is close to π, and by an anisotropic zone if the directions of gradients form an angle close to $\frac{\pi}{2}$. In the last zone the normal to the layers divides the angle between the gradients in half; the optimal medium tries to turn the direction of the vector of heat flow in the appropriate direction.

The drafts of the vector lines ∇T and $\nabla\lambda$ are shown in **??**. It is assumed that the function $\rho(r)$ is defined by (70). All vector lines ∇T begin on the lower face and end on the upper face (**??**); all vector lines $\nabla\lambda$ begin on the part of the lower face where $\rho = 0$ and end on the part where $\rho = 1$. The exterior vector line $\nabla\lambda$ passes around the lateral surface and the upper face of the cylinder, returning to the lower face along the axis Oz (**??**).

It is not difficult to see (**??**) that the zone $\boldsymbol{A}$, where the angle ψ is close to zero, adjoins the lateral surface of the cylinder. The zone $\boldsymbol{C}$ is located near the axes of the cylinder; here this angle is close to $\frac{\pi}{2}$. Zones $\boldsymbol{A}$ and $\boldsymbol{C}$ join at the endpoint of a "window" through which one should pass the maximal quantity of heat; they are divided by zone $\boldsymbol{B}$ where the angle ψ is close to $\frac{\pi}{4}$; this zone adjoins the upper face. In zone $\boldsymbol{B}$, the layered composites are optimal; the directions of the layers are shown by the dotted line.

Physically, the *thermolens* focuses the heat due to the following effects:

1. The heat flow is forced out from zone $\boldsymbol{A}$ by the low-conductivity material σ_1.

2. The heat flow turns in a favorable direction due to refraction in the optimally oriented layers in zone $\boldsymbol{B}$.

3. The heat flow is concentrated in zone $\boldsymbol{C}$, which is occupied by the high-conductivity material σ_2.

The next example of "inhomogeneous heater" demonstrates construction that maximize the temperature in a target point. It was obtained in [?] The problem is similar to the previous one.

Example 4.3 Consider a domain ($\Omega: \ 0 < x < 1,\ 0 < y < 1$) filled with two materials with heat conductivities σ_1 and σ_2. Suppose that the boundary $\partial\Omega$ is kept at zero temperature and that the domain has a concentrated source inside.

The equilibrium is described by the boundary value problem

$$\begin{array}{ll} \boldsymbol{j} = \sigma(x,y)\nabla T, \quad \nabla\cdot\boldsymbol{j} = \delta(x-x_0, y-y_0), & \text{in } \Omega \\ T = 0, & \text{on } \partial\Omega \end{array}$$

Here T is the temperature;

$$\boldsymbol{\sigma}(x,y) = \chi(x,y)\sigma_1 + (1-\chi_1(x,y))\sigma_2;$$

$\delta(x-x_0, y-y_0)$ is the δ-function supported at the point (x_0, y_0) where the source is applied.

It is required to lay out the materials in the domain to maximize the temperature $T(x_t, y_t)$ at a target point $(x_t, y_t) \in \Omega$. The functional is

$$I = \int_\Omega T(x,y)\delta(x-x_t, y-y_t)dx\,dy.$$

?? demonstrates the directions of ∇T and $\nabla\lambda$. **??** shows the optimal project.

The optimal layout provides the best conductance between the source and target points and also insulate the boundary. Notice the optimal layout in the proximity of the source and target points: the volume fraction of the insulator varies from zero (in the direction between the point) to one (in the opposite direction).

5 Conclusion and Problems

Relaxation and G-Closure We already mentioned that the solution to an optimal design problem exists if the set of controls is G_m-closures. If this set is known, one could choose the element $\boldsymbol{\sigma}_* \in G_m U$ that provides the minimum of the functional J. However, here we have shown an alternative, straightforward way of solution, so that unnecessary difficulties of a complete description of G_m-closure are avoided.

Recall that the problem of optimal conductivity of a composite was solved using only the simplest bounds of the G_m-closure or by using the weak G-closure (Chapter 3). We deal with a pair of fields $\boldsymbol{p}$ and $\boldsymbol{q}$, and we minimize their weighted scalar product $\boldsymbol{p}\cdot\boldsymbol{\sigma}_*\boldsymbol{q}$. In this problem, we are looking for a tensor with an extreme element; we are not interested in a description of other elements of it. The extremal tensors belong to a special component of the boundary of $G_m U$; it is enough to describe this component.

Another way to look at this phenomenon is as follows. The constitutive relations in a medium $\langle\boldsymbol{j}\rangle = \boldsymbol{\sigma}_* \langle\boldsymbol{e}\rangle$ are completely determined by the projection of the tensor $\boldsymbol{\sigma}_*$ on a plane formed by vectors $\langle \boldsymbol{j} \rangle$ and $\langle \boldsymbol{e} \rangle$. Particularly, for any optimization problem only this projection is needed, no matter what the cost functional is. These arguments show again that it is enough to describe only a two-dimensional projection on the plane of eigenvectors of maximal and minimal eigenvalues (the weak G-closure). Moreover, only the "corner" of this two-dimensional set is needed to solve the problem.

Clearly, the results obtained could easily be extended to composites of more than two components, which are considered here for simplicity. The result in the general case is the same: Optimal structures are just laminates that bisect the directions of the fields $\langle \boldsymbol{p} \rangle$ and $\langle \boldsymbol{q} \rangle$.

We also could apply this method of relaxation to optimization problems for the processes associated with more general operators than the conductivity operators. The main qualitative result remains the same: The optimal medium has the maximal difference between the energies stored in two different fields. However, we would observe that Wiener bounds are not achievable by a laminate structure. Our next goal, therefore, is to develop a technique for strict bounds and to enlarge the class of available microstructures.

Problems

1. Consider the problem of Section 5.2 for an anisotropic material with variable orientation of the eigenvectors. The G-closure corresponds to all polycrystals. Using the general properties of G-closures, show that the range of eigenvalues of polycrystals lies inside the range of eigenvalues of the original anisotropic material. Analyze the necessary conditions. Do microstructures (polycrystals) appear in an optimal project?

2. Consider the problem of Section 5.2 for three conducting materials. Analyze the necessary conditions. What microstructures are optimal? Does the material with the intermediate conductivity appear in an optimal project?

3. Consider the problem of Section 5.2 with the additional constraint

$$\int_\Omega \chi = M.$$

   Derive and analyze the necessary conditions. Can the Lagrange multiplier associated with the constraint change its sign? When does the multiplier equal zero?

4. Investigate the problem of Section 5.2 about the process described by the differential equation

$$\nabla \cdot \sigma(\chi)(\nabla w + \boldsymbol{\phi}(w)) + \boldsymbol{v} \cdot \nabla w + k(\chi) w = f,$$

   where $\boldsymbol{\phi}, k, f$ are the differentiable functions, and $\boldsymbol{v}$ is a given vector field. Derive the differential equation for the Lagrange multiplier. What microstructures are optimal? What is the relaxed form of the problem?

5. Consider the example of Section 5.4. Assume that the boundary conditions on the lateral surface are

$$w(h,1) - \kappa \frac{\partial w(h,r)}{\partial r}\bigg|_{r=1} = 0.$$

Derive the boundary conditions for λ and discuss the dependence of the optimal structure on κ.

6. Qualitatively describe a composite in an optimal conducting rectangular plate $-1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1$, subject to the boundary conditions

$$\frac{\partial w}{\partial x_2}\bigg|_{x_1=-1} = \frac{\partial w}{\partial x_2}\bigg|_{x_1=1} = 0, \quad w|_{x_2=-1} = 0, \quad w|_{x_2=1} = 1$$

with respect to the following optimality requirements:

(a) A domain in the center is kept at lowest temperature

$$\min \int_{\Omega_\varepsilon} T,$$

where Ω_ε is a circle with center at $(0,0)$ and with the radius much smaller than one. Derive the problem for Lagrange multiplier λ and draw a draft of the gradient lines of ∇w and $\nabla \lambda$. Draw a draft of the optimal layout.

(b) The current $\boldsymbol{j}$ in Ω_ε is directed to minimize the functional

$$\min \int_{\Omega_\varepsilon} \boldsymbol{j} \cdot \boldsymbol{t}, \quad \boldsymbol{t} = \text{constant}.$$

Derive the equation for the Lagrange multiplier λ and draw a draft of ∇w and $\nabla \lambda$. Draw a draft of the optimal layout.

Hint. To draw the draft of an optimal project, sketch the fields ∇w and $\nabla \lambda$ for the isotropic domain and apply necessary conditions to find an approximation to the optimal layout.

6 Compliment 1. Relaxation and G-Convergence

6.1 Weak Continuity and Weak Lower Semicontinuity

First, we describe the type of functionals that can be minimized by homogenization methods. In minimizing the functional, we likely end up with a highly inhomogeneous sequence of materials layout. The homogenization replaces these highly inhomogeneous materials with effective materials. The question is: How does this replacement effect a cost functional?

Formulation Consider again a domain Ω with a smooth boundary $\partial\Omega$ filled with a two-phase inhomogeneous material of conductivity $\sigma(\chi)$, where $\chi = \chi(\boldsymbol{x})$ is the characteristic function of the subdomain Ω_1 occupied with the material σ_1. The rest of the domain is filled with material σ_2.

Consider the conductivity equations (**??**), (**??**), (**??**), and (**??**) in Ω. We rewrite them for convenience:

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{j} &= q \\ \sigma(\chi)\nabla w &= \boldsymbol{j} \end{aligned}\right\} \text{ in } \Omega, \qquad \begin{aligned} w &= \rho_1 \text{ on } \partial\Omega_1, \\ j_n &= \rho_2 \text{ on } \partial\Omega_2, \end{aligned} \tag{73}$$

where w is a potential, $\boldsymbol{j}$ is the current, q is the density of sources, ρ_1 and ρ_2 are the boundary data imposed on the supplementary components $\partial\Omega_1$, $\partial\Omega_2$ of the boundary $\partial\Omega$, and $j_n = \boldsymbol{j} \cdot \boldsymbol{n}$ is the normal component of $\boldsymbol{j}$. We assume that $\rho_1(s)$, $\rho_2(s)$, and $q(\boldsymbol{x})$ are differentiable, where s is the coordinate on the surface $\partial\Omega$.

We suppose that problem (73) has a unique solution for the given q, ρ_1, ρ_2 and for arbitrary layout χ, and this solution is bounded,

$$\|w\|_{H^1(\Omega)} = \int_\Omega (\nabla w^2 + w^2) < C.$$

Consider the minimization problem

$$\min_{w \text{ as in } (73)} I(w); \quad I(w) = \int_\Omega F(w, \nabla w), \tag{74}$$

where w is the solution to (73). Equation (73) is treated as a constraint on $w(\chi)$, and χ is the control. Therefore, the functional I is determined by the control, too: $I = I(w(\chi))$. We denote by $\mathcal{U}$ the set of conductivities of the available materials. Thus we have formulated a restricted variational problem (the so-called Mayer-Bolza problem; see, for example, [**?**]).

The scheme of the dependence of the functional on the control is as follows:

$$\chi \quad \Longrightarrow \quad w(\chi) \quad \Longrightarrow \quad I(w).$$

Stability against Homogenization In dealing with structural optimization problems we expect that a minimizer $\chi(\boldsymbol{x})$ is characterized by fine-scale oscillations. Homogenization can be used to describe such oscillatory solutions. Let us find a class of functionals that can be minimized by this approach. The corresponding mathematical technique is the theory of sequentially weak lower semicontinuity of functionals. We give an informal introduction to the use of this theory in homogenization, and we refer the reader to a rigorous exposition in [**?**, **?**, **?**, **?**].

Compare the potential w_ε associated with the problem for the conductivity operator $\nabla\sigma_\varepsilon\nabla$ with fast oscillating coefficients σ_ε and the potential w_0 associated with the homogenized conductivity operator $\nabla\sigma_*\nabla$ with smooth coefficients σ_*. Recall that w_ε tends to w_0,

$$\lim_{\varepsilon\to 0} \int_\Omega (w_\varepsilon - w_0)^2 = 0, \quad \text{or} \quad w_\varepsilon \to w_0 \text{ strongly in } L_2. \tag{75}$$

However, ∇w_0 and ∇w_ε are not close pointwise:

$$\lim_{\varepsilon \to 0} \int_\Omega (\nabla w_\varepsilon - \nabla w_0)^2 > 0 \tag{76}$$

because ∇w_ε is a discontinuous function that has finite jumps on the boundary of the regions of different materials and ∇w_0 is a continuous function. The limit ∇w_0 represents the mean value of ∇w_ε over an arbitrary regular small region Ω_ε when the frequency of oscillations $\frac{1}{\varepsilon}$ goes to infinity,

$$\nabla w_0 = \lim_{\varepsilon \to 0} \langle \nabla w_\varepsilon \rangle; \tag{77}$$

∇w_ε weakly converges to ∇w_0 in L_2.

Weakly Continuous Functionals Some functionals $I(w)$ are stable under homogenization, while others can significantly change their value: $I(w_0) - I(w_\varepsilon) = 0(1)$. In the latter case, the solution to the homogenized problem may have nothing in common with the solution to the original problem. Therefore, it is important to distinguish these types of functionals.

The functional $I(w)$ is called *weakly continuous*, if

$$I(w_0) = \lim_{w_s \rightharpoonup w_0} I(w_s) \tag{78}$$

where $\rightharpoonup$ means the weak convergence in $H^1(\Omega)$. Weakly continuous functionals are stable under homogenization.

For example, the functional

$$I_1(w) = \int_\Omega (F(w) + \boldsymbol{A} \cdot \nabla w),$$

is weakly continuous if F is a continuous function and $\boldsymbol{A}$ is a constant or smoothly varying vector. Indeed, (75), (77) imply that

$$I_1(w_\varepsilon) - I_1(w_0) = \int_\Omega (F(w_\varepsilon) - F(w_0)) + \int_\Omega (\boldsymbol{A} \cdot \nabla(w_\varepsilon - w_0)) \to 0 \tag{79}$$

when $\varepsilon \to 0$. The first integral goes to zero because $\|w_0 - w_\varepsilon\|_{L_2}$ is arbitrary near to zero and the function F is continuous, and the second integral goes to zero because ∇w_ε goes to ∇w_0 weakly in L_2. Therefore, $I_1(w)$ is stable under homogenization.

Generally, the functional $I(w) = \int_\Omega F(\nabla w)$, where F is a nonlinear function, does not keep its value after homogenization. The limit

$$I^0 = \lim_{w_s \rightharpoonup w_0} I(w_\varepsilon),$$

where w_s is a fine-scale oscillatory function, can generally be either greater or smaller than $I(w_0)$ depending on the minimizing sequence $\{w_s\}$. These functionals are called *weakly discontinuous*.

Weakly Lower Semicontinuous Functionals The weak continuity is sufficient but not necessary to pass to the weak limit of the minimizer in the variational problem. The property needed is called the *weak lower semicontinuity* [**?**]:

$$I^0 = \lim_{\varepsilon \to 0} I(w_\varepsilon) \geq I(w_0), \qquad w_\varepsilon \rightharpoonup w_0. \tag{80}$$

For these functionals, the limit only decreases when the minimizer coincides with the weak limit of the minimizing sequence.

Each functional of the type

$$I_2(w) = \int_\Omega F(w, \nabla w), \tag{81}$$

where $F(w, \nabla w)$ is a continuous function of w and a convex function of ∇w, is weakly lower semicontinuous.

Example 6.1 The functional

$$I_3(w) = \int_\Omega (\nabla w)^2 \tag{82}$$

is weakly lower semicontinuous, because

$$I_3^0 = \lim_{w_\varepsilon \rightharpoonup w_0} I(w_\varepsilon) = I_3(w_0) + \int_\Omega (\nabla w_\varepsilon - \nabla w_0)^2 \geq I_3(w_0),$$

when $w_\varepsilon \rightharpoonup w_0$ (to compute I_3^0 we use the limit $\nabla(\langle w_\varepsilon \rangle - w_0) \rightharpoonup 0$).

Dependency on χ A broader class of optimization problems deals with functionals

$$I_4(w, \chi) = \int_\Omega F(w, \nabla w, \chi) \tag{83}$$

that explicitly depend on the characteristic function χ. The functional I_4 depends on the amount (or cost) of the materials used in the design. For these problems, weak lower semicontinuity is formulated as

$$\lim_{\varepsilon \to 0} I_4(w_\varepsilon, \chi_\varepsilon) \geq I(w^0, m), \quad \text{when} \begin{cases} \chi_\varepsilon & \rightharpoonup & m, \\ w_\varepsilon & \rightharpoonup & w_0. \end{cases} \tag{84}$$

Analogously to (81), the functional I_4 is weakly lower semicontinuous if $F(w, \nabla w, \chi)$ is a continuous function of w and a convex function of ∇w and χ.

Remark 6.1 Dealing with relaxation of the nonconvex Lagrangians, we consider the inverse problem: What property of the Lagrangian is necessary and sufficient for the weakly lower semicontinuity? This question is discussed in the next chapter. Here, we only mention that the convexity of $F(w, \nabla w, \chi)$ with respect to ∇w and χ and the continuity with respect to w is sufficient for the weakly lower semicontinuity.

In summary, the homogenization technique is directly applicable to the weakly lower semicontinuous functionals that do not increase their values by homogenization.

6.2 Relaxation of Constrained Problems by G-Closure

G-Closed Sets of Control Here we are dealing with a variational problem with differential constraints that expresses the equilibrium (the Mayer-Bolza problem): w is the solution of an equilibrium problem defined by the control.

Consider the minimization of a weakly lower semicontinuous functional $I(w)$, where w is the solution to a boundary value problem (73). Consider a sequence of solutions $w_\varepsilon = w(\sigma_\varepsilon)$ that minimizes the functional $I(w)$:

$$I(w_\varepsilon) \ \to \ I_0 = \inf I(w). \tag{85}$$

Because the functional is weakly lower semicontinuous, the minimizing sequence $\{w_\varepsilon\}$ weakly converges to w_0,

$$w_\varepsilon \rightharpoonup w_0 : \quad I_0 = I(w_0). \tag{86}$$

Let us find out what happens to the corresponding sequence of materials' layouts σ_ε. Recall the definition of G-convergence (Chapter 3): A sequence $\{\sigma_\varepsilon\}$ G-converges to a tensor-valued function $\boldsymbol{\sigma}_*$ if the sequence $\{w_\varepsilon\}: \ w_\varepsilon = w(\sigma_\varepsilon)$ weakly converges to $w_0 = w(\boldsymbol{\sigma}_*)$. Therefore, an optimal solution w_0 (a weak limit of w_ε) corresponds to a G-limit $\boldsymbol{\sigma}_*$ of $\{\sigma_\varepsilon\}$:

$$I_0 = \inf_{\sigma_\varepsilon} I(w(\sigma_\varepsilon)) = \min_{\boldsymbol{\sigma}_* \in GU} I(w(\boldsymbol{\sigma}_*)). \tag{87}$$

A minimization problem of a weakly lower semicontinuous functional has a solution in the set $\mathcal{U}$ of values of σ if all G-limits $\boldsymbol{\sigma}_*$ belong to this set or if the set $\mathcal{U}$ is G-closed.

Remark 6.2 This approach, called the "homogenization approach," was developed in different forms in many papers [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] We do not mention here the control problems dealing with ordinary differential equations, where these ideas were developed several decades earlier.

G-Closure of the Set of Controls We mentioned in Chapter 3 that the set of conductivities $\mathcal{U}$ usually does not coincide with its G-closure. In that case, the solution to problem (74) may not exist. This means that any value $I(w^s)$ that corresponds to a layout of materials $\sigma(\chi^s)$ can be decreased by another layout $\sigma(\chi^{s+1})$ and so on.

Example 6.2 Recall the simplest example of the absence of an optimal element in an open set; there is no minimal positive number. The infimum of all positive numbers is zero, but zero is not included in the set. To make the problem of the minimal positive number well-posed, one must enlarge the set of positive numbers by including the limiting point (zero) in it. Similarly, we reformulate (relax) the optimal design problem including all G-limits into the set of admissible controls. An extended problem has a solution equal to the limit of the minimizing sequences of solutions to the original ill-posed problems.

A functional $I_4(w,\ \chi)$ (84) that explicitly depends on χ requires similar relaxation. This time we need to consider the convergence of a pair $(w^s,\ \chi^s)$ to the pair $(w_0,\ m)$, where $m = m(\boldsymbol{x})$ is the variable volume fraction of the first material in an optimal composite. Accordingly, the G_m-closures (instead of G-closures) are used for the relaxation.

Notion of G-Closure is Sufficient (but Not Necessary) for Relaxation The problem of existence of optimal controls for lower weakly continuous functionals is trivially solved when one passes from the set $\mathcal{U}$ to its G- or G_m-closure. However, this problem is in fact replaced by another one: how to find the G_m-closure itself. The last problem is by no means simpler than the problem of relaxation. On the contrary, some optimization problems are less complex than the problem of the description of the G-closure. The problem in Chapter 4 is an example of relaxation without complete description of the G_m-closure.

More direct approaches to the relaxation of the optimal problems than the G-closure procedure have been considered in a number of papers, beginning with [**?**, **?**, **?**, **?**, **?**]. It was noted that the given values of the pair of current and gradient fields in the constitutive relations do not determine the tensor of properties completely. One can consider the equivalence of the class of anisotropic tensors [**?**] that produces the same current in a given gradient field. This idea (the weak G-closure; see Chapter 3) allows one to prove that only laminates can be used for relaxation.

A different approach [**?**, **?**] is based on direct bounds of the value of the minimax augmented functional that replaces the original minimal problem with differential constraints. It was suggested to use the saddle-type Lagrangians for the upper and lower bound of the min-max variational problem for an augmented functional. That approach enables one to immediately find the bounds for a minimax problem.

The Detection of Ill-Posed Extremal Problems To find whether the optimization problem is ill-posed and needs relaxation, one can use the Weierstrass test. For example, one can consider the variation in a strip and detect the existence of a forbidden region where the test fails. This approach was suggested in early papers [**?**, **?**] where the Weierstrass-type conditions were derived and the forbidden region was detected in a problem similar to the one considered here.