Euler equations for multiple integrals

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1 Euler equation

Consider the simplest problem of multivariable calculus of variation: Minimize an integral of a twice differentiable Lagrangian $F(x, u, \nabla u)$ over a regular bounded domain Ω with a smooth boundary $\partial\Omega$. The Lagrangian F depends on the minimizer u and its gradient ∇u with the function u taking prescribed values u_0 on the boundary $\partial\Omega$,

$$\min_{u:\, u|_{\partial\Omega}=u_0} I(u), \quad I(u) = \int_\Omega F(x, u, \nabla u)dx \tag{1}$$

As in the one-variable version (see Chapter 2), the Euler equation expresses the stationarity of the functional I with respect to the variation of u. To derive the Euler equation, we consider the variation δu of the minimizer u and the difference $\delta I = I(u + \delta u) - I(u)$. We assume that for any x, the variation δu is localized in an ϵ-neighborhood of point x, twice differentiable, and small: the norm of its gradient goes to zero if $\epsilon \to 0$,

$$\delta u(x + z) = 0,\ \forall z:\ |z| > \epsilon, \quad |\nabla(\delta u)| < C\epsilon,\ \forall x \tag{2}$$

For any minimizer u, the difference δI must be nonnegative, $\delta I(u, \delta u) \geq 0\ \forall \delta u$.

Increment When the variation δu and its gradient are both infinitesimal and F is twice differentiable, we can linearize the perturbed Lagrangian:

$$\begin{aligned} F(x, u + \delta u, \nabla(u + \delta u)) &= F(x, u, \nabla u) + \frac{\partial F(x, u, \nabla u)}{\partial u}\delta u \\ &+ \frac{\partial F(x, u, \nabla u)}{\partial \nabla u}\delta \nabla u + o(||\delta u||, ||\nabla \delta u||) \end{aligned}$$

Here, the term $\frac{\partial F(x,u,\nabla u)}{\partial \nabla u}$ denotes the vector of the partial derivatives of F with respect to partial derivatives of u,

$$\frac{\partial F(x, u, \nabla u)}{\partial \nabla u} = \left[\frac{\partial F(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_1}\right)}, \ldots, \frac{\partial F(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_n}\right)} \right].$$

Substitution of the linearized Lagrangian into the expression for δI results in the following expression:

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u}\delta u + \underline{\frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u} \right) dx + o(||\delta u||, ||\nabla \delta u||).$$

Next, we transform the underlined term. Interchanging two linear operators of variation and differentiation, $\delta \nabla u = \nabla \delta u$, and performing integration by parts (see (**??**)), we obtain

$$\int_\Omega \left(\frac{\partial F}{\partial \nabla u} \cdot \nabla(\delta u) \right) dx = -\int_\Omega \delta u \left(\nabla \cdot \frac{\partial F}{\partial \nabla u} \right) dx + \int_{\partial\Omega} \delta u \left(\frac{\partial F}{\partial \nabla u} \cdot n \right) ds$$

so that

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u} - \nabla \cdot \frac{\partial F}{\partial \nabla u} \right) \delta u \, dx + \int_{\partial\Omega} \delta u \left(\frac{\partial F}{\partial \nabla u} \cdot n \right) \, ds$$

The coefficient by δu in the first integral is called the *variational derivative in* Ω or the *sensitivity function*:

$$S_F(u) = \frac{\partial F}{\partial u} - \nabla \cdot \left(\frac{\partial F}{\partial \nabla u} \right) \tag{3}$$

The coefficient by δu in the boundary integral is called the *variational derivative on the boundary* $\partial\Omega$:

$$S_F^\partial(u, n) = \frac{\partial F}{\partial \nabla u} \cdot n = \frac{\partial F}{\partial \left(\frac{\partial u}{\partial n} \right)} \tag{4}$$

Hence, we represent the linearized increment δI as a sum of two terms:

$$\delta I = \int_\Omega S_F(u) \delta u \, dx + \int_{\partial\Omega} S_F^\partial(u, n) \delta u \, ds. \tag{5}$$

Stationarity The condition $\delta I \geq 0$ and the arbitrariness of variation δu in the domain Ω and possibly on its boundary $\partial\Omega$ leads to the stationarity condition in a form of differential equation:

$$S_F(u) = 0 \quad \text{or} \quad - \nabla \cdot \frac{\partial F}{\partial \nabla u} + \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega \tag{6}$$

with the boundary condition

$$S_F^\partial(u, n) \, \delta u = 0 \quad \text{on } \partial\Omega \tag{7}$$

Equation (6) with the boundary condition (7) is the **Euler-Lagrange equation** for variational problems dealing with multiple integrals. Notice that we keep δu in the expression for the boundary condition. This allows us to either assign u on the boundary or leave it free, which corresponds to two different types of boundary condition.

The main boundary condition In the considered simplest problem, the partial differential equation (6) is given in Ω with the boundary conditions $u = u_0$. The boundary term (7) of the increment vanishes because the value of u on the boundary is prescribed, hence the variation δu is zero. This condition is called the *main boundary condition*. It is assigned independently of any variational requirements. When u is prescribed on some component of the boundary, we say that the main boundary condition is posed; in this case the variation of u on this part of the boundary is zero, $\delta u = 0$, and (7) is satisfied.

Natural boundary condition If the value of u on the boundary is not specified, the term (7) supplies additional boundary condition. If no condition is prescribed on a boundary component, δu is an arbitrary function, and the natural condition

$$S_F^{\partial}(u,n) = \frac{\partial F}{\partial(\nabla u)} \cdot n = 0 \quad \text{or} \quad \frac{\partial F}{\partial u_n} = 0 \tag{8}$$

(see (**??**)) must be satisfied. Notice, that the natural boundary condition follows from the minimization requirement; it must be satisfied to minimize the functional in (1).

Thus the boundary value problem in the domain Ω has one condition, main or natural, on each component of the boundary.

Remark 1.1 Notice that the stationarity and the natural boundary conditions are analogous to corresponding conditions for a one-variable Euler equation. The derivative $\frac{d}{dx}$ with respect to the independent variable is replaced by ∇ or by $\nabla\cdot$. At the boundary, the derivative $\frac{du}{dx}$ is replaced by $\frac{\partial u}{\partial n} = \frac{\partial}{\partial \nabla u} \cdot n$. In the last case, the derivative with respect to x becomes the directional derivative along the normal to the boundary.

Remark 1.2 The existence of a solution to the boundary value problem (6), (7) in a bounded domain Ω requires ellipticity of the Euler equation. In turn, the ellipticity imposes some requirements on properties of the Lagrangian; this will be discussed later.

2 Examples of Euler-Lagrange equations

Here, we give several examples of Lagrangians, the corresponding Euler equations, and natural boundary conditions. We do not discuss the physics and do not derive the Lagrangians from general principles of symmetry; this will be done later. Here, we formally derive the stationary equations.

Example 2.1 (Laplace's equation) Consider a Lagrangian quadratically depending on ∇u:

$$F = \frac{1}{2}\nabla u \cdot \nabla u$$

This Lagrangian corresponds to the energy of a linear conducting medium of unit conductivity. We compute the variational derivative of F:

$$\frac{\partial F}{\partial \nabla u} = \nabla u, \quad S_F(u) = -\nabla \cdot \frac{\partial F}{\partial \nabla u} = -\nabla \cdot \nabla u.$$

The stationarity condition or the Euler equation, $S_F(u) = 0$, is Laplace's equation:

$$-\nabla \cdot \nabla u = -\Delta u = 0$$

where (-Δ) is the Laplace operator, or the Laplacian. In the coordinate notation, Laplace's equation has the form:

$$S_F(u) = -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

The natural boundary condition is

$$S_F^{\partial}(u, n) = \nabla u \cdot n = \frac{\partial u}{\partial n} = 0$$

Notice, that if no values of function u is prescribed on the boundary, so that the natural boundary condition is posed on whole boundary $\partial\Omega$, then identically zero solution, $u = 0$, is the only solution of the variational problem. To get a non-trivial solution, the main boundary condition should be given on a part of the boundary.

Example 2.2 (Linear elliptic equation) We consider a more general Lagrangian corresponding to the energy density of a linear conducting heterogeneous anisotropic material:

$$F = \frac{1}{2}\nabla u \cdot A(x)\nabla u$$

Here $A(x) = \{A_{ij}(x)\}$ is a symmetric positively defined conductivity tensor that represents the material properties, and u is the potential such as temperature, electric potential, or concentration of particles. The steady state distribution of the potential minimizes the total energy or solves the variational problem (1) with the Lagrangian F. We comment on the derivation of this energy below in Section **??**. Here, we are concerned with the form of the stationarity condition for this Lagrangian. The variational derivative is the following:

$$\frac{\partial F}{\partial \nabla u} = A\nabla u, \quad S_F(u) = -\nabla \cdot \frac{\partial F}{\partial \nabla u} = -\nabla \cdot A\nabla u.$$

The stationarity condition (Euler equation) is the second-order elliptic equation:

$$S_F(u) = -\nabla \cdot A(x)\nabla u = 0$$

which in the coordinate notation, has the form:

$$S_F(u) = -\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} = 0.$$

The natural boundary condition is $S_F^{\partial}(u, n) = A\nabla u \cdot n = 0$.

When A is proportional to the unit matrix I, $A = \kappa(x)I$, where $\kappa > 0$ is the scalar conductivity, the Lagrangian becomes

$$F = \frac{\kappa(x)}{2}\nabla u \cdot \nabla u$$

The corresponding Euler equation

$$\nabla \cdot \kappa(x) \nabla u = 0$$

describes conduction process in a inhomogeneous isotropic medium with a spatially varying scalar conductivity function $\kappa(x) > 0$. Using coordinate notation, the equation is written as:

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} k(x) \frac{\partial u}{\partial x_i} = 0$$

The natural boundary condition is called the *homogeneous Neumann condition*:

$$k(x) \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

it can be simplified to $\frac{\partial u}{\partial n} = 0$ if $k(x)) > 0$.

In the general case of anisotropic conductivity given by a tensor A, the natural boundary condition corresponding to the *homogeneous Neumann condition* is:

$$A(x) \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

Notice that in this case, the directional derivative, not the normal derivative, is zero on the boundary.

The main boundary condition is called the *Dirichlet boundary condition*:

$$u = u_0 \quad \text{on} \quad \partial\Omega$$

Example 2.3 (Poisson and Helmholtz equations) These classical linear elliptic equations of mathematical physics originate from a variational problem of minimization of a quadratic Lagrangian. The Lagrangian of a form:

$$F = \frac{1}{2}|\nabla u|^2 - \frac{1}{2} a u^2 - bu \tag{9}$$

corresponds to the Euler equation $S_F(u) = 0$:

$$\Delta u + au + b = 0$$

which is called the inhomogeneous Helmholtz equation. The natural boundary condition $\frac{\partial u}{\partial n} = 0$ is independent of a and b. If $a = 0$, the inhomogeneous Helmholtz equation degenerates into *Poisson equation*. If $b = 0$, it becomes homogeneous *Helmholtz equation*, and if $a = b = 0$ it degenerates into Laplace equation.

Example 2.4 (Nonlinear elliptic equation) Assume that the Lagrangian depends only on magnitude of the gradient:

$$F = \phi(|\nabla u|) \tag{10}$$

where ϕ is a monotonically increasing convex function, $\phi'(z) > 0$, $\forall z \in [0, \infty)$. Such Lagrangians describe the steady state conductivity or diffusion process in an isotropic nonlinear medium; u is the potential or concentration of diffusing particles.

Let us assume that $|\nabla u|$ does not turn to zero. The Euler equation is computed as

$$\nabla \cdot (\kappa(|\nabla u|)\nabla u) = 0, \quad \kappa(z) = \frac{\phi'(z)}{|z|}$$

Since $\phi' > 0$, the equation is elliptic. It also can be rewritten as two first-order equations

$$\nabla \cdot j = 0, \qquad j = \phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}$$

where j is a divergencefree *current* vector. The first equation express the equilibrium of the current density. The second equation is called the *constitutive relation*. It demonstrates the property of the material and characterizes the dependence of the current on the *field* ∇u. The coefficient $\frac{\phi'(|\nabla u|)}{|\nabla u|}$ is the conductivity of a nonlinear material; it depends on the magnitude of the field.

The natural boundary condition is

$$\frac{\phi'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial n} = 0$$

Because $\phi' > 0$, it simplifies to $\frac{\partial u}{\partial n} = 0$ and again expresses the vanishing of the normal derivative of u on the boundary.

In the next examples, we specify the function ϕ and obtain the variational form of well-studied nonlinear equations.

Example 2.5 (Nonlinear elliptic equation) The previous problem simplifies if the Lagrangian depends on the squared magnitude of the gradient:

$$F = \psi(|\nabla u|^2) \tag{11}$$

we assume again here that ψ is a monotonically increasing convex function. Differentiating F, we have:

$$\frac{\partial F}{\partial \nabla u} = 2\psi'(|\nabla u|^2)\nabla u$$

So that the Euler equation is

$$\nabla \cdot \left(\psi'(|\nabla u|^2)\nabla u\right) = 0$$

The conductivity of the nonlinear medium in this case, is $\psi'(|\nabla u|^2)$; as in the previous case, it depends on the magnitude of the field. Special case, when $\psi(z) = z$ results in Laplace equation.

Example 2.6 (p-Laplacian) Consider the Lagrangian that corresponds to special nonlinearity $\phi(z) = \frac{1}{p}z^p$ in (10)

$$F = \frac{1}{p}|\nabla u|^p \tag{12}$$

The Euler equation is:

$$\nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0$$

The equation is called p-Laplacian. It degenerates into Laplace equation when $p = 2$.

Example 2.7 **($p = 1$)** Another interesting case $p = 1$. The Lagrangian becomes the norm of the gradient,

$$F = |\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2} \tag{13}$$

(here, we consider for the definiteness the two-dimensional case). The corresponding Euler equation is:

$$\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \quad \text{in } \Omega$$

In this case, the isotropic nonlinear conductivity function is $|\nabla u|^{-1}$. Here again, we assume that $|\nabla u| \neq 0$, otherwise $|\nabla u|$ can be approximated as $|\nabla u| = \sqrt{|\nabla u|^2 + \beta}$ for some small parameter β.

Similar to the general case, the Euler equation can be written as a system of two first-order partial differential equations

$$j = \frac{\nabla u}{|\nabla u|}, \quad \nabla \cdot j = 0.$$

Observe that in this case $|j| = 1$ Here, the current j is codirected with ∇u and has the unit magnitude. In 2D case, any unit vector admits the representation

$$j = (j_1, j_2), \quad j_1 = \cos\theta, \quad j_2 = \sin\theta$$

where $\theta(x)$ is an unknown scalar function, that is defined by the first-order equation $\nabla \cdot j = 0$ or

$$-\sin\theta \frac{\partial \theta}{\partial x_1} + \cos\theta \frac{\partial \theta}{\partial x_2} = 0$$

Potential u is found from another first-order equation that states that j is parallel to ∇u, or $j \times \nabla u$=0. In the coordinate form, the equation becomes

$$\frac{\partial u}{\partial x_1} j_2 - \frac{\partial u}{\partial x_2} j_1 = 0 \quad \text{or} \quad \frac{\partial u}{\partial x_1} + (\cot\theta) \frac{\partial u}{\partial x_2} = 0$$

Notice Lagrangian is not strongly convex function of ∇u and Euler equation is split into two first order equations.

3 Smooth approximation and continuation

As a first application of the multivariable extremal theory, consider a problem of approximation of a given scalar function f of several variables by a function u with assumed smoothness. The problem of approximation of a bounded, integrable, but may be discontinuous function $f(x)$, with x being in some subdomain of R^3, by a smooth function $u(x)$ results in the variational problem

$$\min_u \frac{1}{2} \int_{R^3} \left((u-f)^2 + \epsilon^2 |\nabla u|^2\right) dx$$

where the term $\epsilon^2|\nabla u|^2$ represents penalization. If $\epsilon \ll 1$, the first term of the integrand prevails, and u accurately approximates f. As the parameter ϵ grows, the approximation becomes less accurate but the function u becomes more smooth. When $\epsilon \gg 1$, the approximation u tends to a constant function equal to the mean value of f.

The Euler equation for the approximation u is the inhomogeneous Helmholtz equation:

$$\epsilon^2 \nabla^2 u - u = -f, \qquad \lim_{|x|\to\infty} u(x) = 0$$

This inhomogeneous Helmholtz problem can be explicitly solved using Green's function representation. In 3D case, we have:

$$u(x) = \int_{R^3} f(y) K(x-y) dy$$

Here $K(x-y)$ is the Green's function which satisfies the equation:

$$(\epsilon^2 \nabla^2 - 1) K(r) = -\delta(r), \qquad \lim_{|r|\to\infty} u(r) = 0$$

The Green's function for this Helmholtz problem for the whole R^3 is

$$K(r) = \frac{1}{4\epsilon^2 \pi |r|} \exp\left(-\frac{|r|}{\epsilon}\right), \quad |r| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Using this representation, we obtain expression for u

$$u(x) = \frac{1}{4\epsilon^2 \pi} \int_{R_3} \exp\left(-\frac{|x-y|}{\epsilon}\right) \frac{f(y)}{|x-y|} dy$$

One observes that the smoothness of u is controlled by ϵ. When $\epsilon \to 0$, the kernel $K(r)$ tends to the delta-function, and $u(x) \to f(x)$.

Remark 3.1 Similar explicit solutions can be derived for R_2 and for some bounded domains, such as rectangles or circles (spheres). Considering the approximation problem on a bounded domain, a more efficient way to construct solution is to use the eigenfunction expansion, as it was explained in Section **??**.

Example 3.1 (Analytic continuation) A close problem is the analytic continuation. Let $\Omega \subset R_2$ be a domain in a plane with a differentiable boundary $\partial\Omega$. Let $\phi(s)$ be a differentiable function of the point s of $\partial\Omega$. Consider the following problem of *analytic continuation*: Find a function $u(x)$ in Ω such that it coincides with ϕ on the boundary, $u(s) = \phi(s), \forall s \in \partial\Omega$ and minimizes the integral of $(\nabla u)^2$ over Ω. Thus, we formulate a variational problem:

$$\min \int_\Omega (\nabla u)^2 dx \ \text{ in } \Omega, \quad \text{subject to} \quad u|_{\partial\Omega} = \phi$$

Compute the stationarity conditions. We have

$$\frac{\partial(\nabla u)^2}{\partial \nabla u} = 2\nabla u, \quad \nabla \cdot \frac{\partial(\nabla u)^2}{\partial \nabla u} = 2\nabla \cdot \nabla u = 2\Delta u = 0$$

which demonstrates that the minimizer must be harmonic in Ω or be a real part of an analytic function. This explains the name "analytic continuation".

Remark 3.2 Notice that the one-dimensional case is trivial: Ω is an interval, the boundary consists of two points, the minimizer is a straight line between these points. In this sense, harmonic functions are two-dimensional (or higher-dimensional) generalization of linear functions.

4 Change of coordinates

In order to transform the variational conditions to polar, spherical, or other coordinates, consider the transformation of the independent variables $x = w(\xi)$ in a multivariable variational problem. Consider the Jacobian of the transformation J, J being the matrix with the elements $J_{ij} = \{\frac{\partial w_i}{\partial \xi_j}\}$, and assume that $\det(J)$ is not zero in all points of Ω. In the new variables, the domain Ω becomes Ω_ξ, the differential dx is transformed as

$$dx = \det(J) d\xi$$

. By the chain rule, gradient $\nabla_x u$ in x coordinates becomes

$$\nabla_x u = \nabla_\xi u \frac{\partial \xi}{\partial x} = J^{-1} \nabla_\xi u$$

where ∇_ξ is the gradient in ξ-coordinates.

The integral

$$R = \int_\Omega F(x, u, \nabla u) dx$$

becomes

$$R = \int_{\Omega_\xi} F_\xi(\xi, u, \nabla_\xi u) d\xi$$

where F_ξ is defined as follows

$$F_\xi(\xi, u, \nabla_\xi u) = F(w(\xi), u, J^{-1}(\xi) \nabla_\xi u) \det J(\xi) \tag{14}$$

The Euler equation in the w-coordinates becomes $S_{F_\xi}(u) = 0$, where

$$S_{F_\xi}(u) = \frac{\partial F_\xi}{\partial u} - \nabla_\xi \cdot \frac{\partial F_\xi}{\partial \nabla_\xi u} \tag{15}$$

and the derivatives are related as

$$\frac{\partial F_\xi}{\partial u} = (\det J)\frac{\partial F}{\partial u} \quad \text{and} \quad \frac{\partial F_\xi}{\partial \nabla_\xi u} = (\det J) J^{-1} \frac{\partial F}{\partial \nabla u}$$

Example 4.1 (Helmholtz equation in polar coordinates) Let F be the Lagrangian corresponding to the Helmholtz equation on the plane with Cartesian coordinates (x, y):

$$F = |\nabla u|^2 + \alpha u^2 = u_x^2 + u_y^2 + \alpha u^2$$

We transform it to the polar coordinates (r, θ) using $x = r\cos\theta$, $y = r\sin(\theta)$ and compute the Euler equation for F. We have

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad \det J = r$$

Then

$$F_\xi = F_\xi(r, \theta, u, \nabla_\xi u) = r\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \alpha\, u^2\right]$$

and the Euler equation becomes

$$\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} - \alpha r u = 0$$

5 First integrals

Independence of the gradient of minimizer If the Lagrangian is independent of ∇u, $F = F(x, u)$, the Euler equation becomes an algebraic relation

$$\frac{\partial F}{\partial u} = 0.$$

As in one-dimensional case, the minimizer u that solves this equation does not need to be differentiable, even continuous function of x.

Independence of the minimizer If the Lagrangian is independent of u, $F = F(x, \nabla u)$ then Euler equation becomes

$$\nabla \cdot \left(\frac{\partial F}{\partial \nabla u}\right) = 0.$$

Instead of the constancy of $\frac{\partial F}{\partial u'}$ in one-dimensional case, here we state only the divergencefree nature of $\frac{\partial F}{\partial \nabla u}$. Any divergencefree vector admit the following representation through a vector potential.

$$\frac{\partial F}{\partial \nabla u} = \nabla \times \psi \tag{16}$$

In the one-variable case, $\nabla \times \psi$ is replaced by a constant and we obtain the first integral; in multivariable case, no additional first integrals exist.

Example 5.1 The Lagrangian $F_1 = \left(\frac{du}{dt}\right)^2, \ t \in R^1$ is a one-dimensional analog of the two-dimensional Lagrangian $F_2 = |\nabla u|^2$. The Euler equation for the one-dimensional problem with this Lagrangian $\frac{d}{dt}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}$, where $u' = \frac{du}{dt}$, has the first integral

$$C_1 = \frac{\partial F_1}{\partial u'} = \frac{du}{dt}$$

followed by a solution $u = C_1 t + C_2$.

In multivariable case, for $F_2 = |\nabla u|^2$ we compute $\frac{\partial F}{\partial \nabla u} = 2\nabla u = V$. Here, we denote the gradient by $V = (v_1, v_2)$, $V = \nabla u$. The stationarity condition $\nabla \cdot V = 0$ or

$$\frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 = 0$$

are identically satisfied if v admits the representation

$$v_1 = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad v_2 = -\frac{\partial \psi}{\partial x_1}$$

where ψ is an arbitrary potential, that is if (16) holds. The function ψ is called the adjoint potential, see below, Section **??**. Instead of being a linear function as in one-dimensional case, the minimizer u is harmonic – a solution to the Laplace equation $\Delta u = 0$. Potential ψ is a conjugate harmonic function.