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The waves of damage in elastic–plastic lattices with waiting links: Design and simulation

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Abstract

We consider protective structures with elastic–plastic links of a special morphology. The links are bistable. They are designed to arrest the development of localized damage (a neck) in a link, and initiate the damage in another sequential link instead. A wave of "partial damage" propagates through the chain, as all the links develop necks but do not fail. A three-dimensional finite element analysis is used to compute the force–elongation relation for the bistable link. Another numerical procedure analyzes the dynamics of a chain or a lattice undergoing a collision with a heavy projectile. The procedure describes the non-linear waves, determines the places of failure and the critical loading conditions. The bistable chain is compared to the chain of conventional links of the same length, mass, and material. When subjected to an impact, the structure absorbs several times more energy than a conventional structure. The bistability leads to an unexpected result: The maximal absorbed energy in the bistable chain is proportional to its volume, not to the cross-section as in the conventional chain. The bistable chain is capable of withstanding a collision with a mass having kinetic energy several times greater. Even when the bistable chain breaks it effectively reduces the speed of the projectile.

Keywords: Protective structure; Elastic–plastic material; Necking; Bistability; Partial damage; Collision; Waves of transition; Damage propagation; Failure

1. Introduction

The failure of an elongated specimen occurs when a localized damaged zone expands across the sample while the large part of the material of a broken sample stays undamaged. It is tempting to design the specimen that uniformly distributes a "partial damage" throughout its volume and utilizes all the material capability. The problem is to create a structure capable of distributing the damage and absorbing the energy of an impact without failure.

The difficulty is the natural localization of the damage: In solid bars, the damaged cross-section is weaker and the material in it is more vulnerable to failure than the material in the undamaged parts. The maximal stress in a solid bar is always located

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near the zones of impact and its supports. On the other hand, a more uniform distribution of the stress can be achieved in a structure with bistable links, which experiences a phase transition: The links transit from initially undamaged state to the partially damaged state. These two locally stable states of a link are separated by an interval of unstable deformation. The transition of a link causes the transition of a neighboring link which initiates a non-linear wave of damage that propagates through the structure. We investigate structures with artificially induced instabilities due to a special morphology of the links that possess non-monotonic constitutive relations. Previously, the waiting link structures were examined in a number of papers. The concept of the structures with waiting links were suggested by Slepyan and Cherkaev (1995). The dynamics of reversible bistable structures were addressed by Balk et al. (2001a,b). The waves in elastic–brittle structures were studied in the papers of Slepyan et al. (2005a,b, 2004), Slepyan and Ayzenberg-Stepanenko (2004). The theory of the waves in unstable materials is considered in the book of Slepyan (2002) and in the papers of Slepyan (2001a,b,c). The dynamics of lattices were simulated by Cherkaev and Zhornitskaya (2004).

In this paper, we deal with bistable structures from the elastic–plastic materials. The model is more sophisticated than the elastic–brittle models that have been examined before. The ANSYS simulation is used to obtain the constitutive relations in a bistable link. The model accounts for the hysteresis of the loading cycle, the realistic geometric parameters and elastic–plastic properties of the specific material. The two simulations are conducted: The first one results in a static constitutive equations and the second one in the dynamics of the phase transition and damage propagation. Each link of the assembly possesses a non-monotonic force–elongation relation; the link can experience an irreversible damage (partial failure) when its equilibrium length increases and elastic modulus changes. It is shown that the waiting link structures sustain a much larger impact than the conventional ones due to the distribution of the partial damage and the increase of the related energy release. The expected applications range from hypersensitive sensors that change their properties in response to a signal, to protective structures, to "dynamic smart materials," and to structures with the controllable damage propagation.

2. Structures with waiting links

2.1. Mechanism of damage propagation

Consider a chain or lattice of bistable links from an elastic–plastic material. Each bistable link consists of the straight *main link* and the longer initially curved *waiting link*. The main and waiting links are made of the same elastic–plastic material and joined at the ends. An example of a one-dimensional chain is shown in Fig. 1.

Initially, the resistance of the assembly is mainly due to the elastic response of the main link. The resistance of the curved waiting link is relatively small. As the elongation increases, the main link enters the plastic zone and develops a neck which reduces its resistance. At that moment, the waiting link is straightened and its resistance sharply increases. The sum of the resistances of the two links becomes bistable, if the parameters of these links are properly chosen: The total resistance increases, then decreases, then increases again. The increasing regimes correspond to the locally stable state, and the decreasing regime to the unstable state. The passage between these states is the *phase transition*.

The properly designed bistable chain is much more resistant than the conventional chain. In a conventional elongated bar of elastic–plastic material, the necking corresponds to the beginning of a fatal instability; the damage is accumulated in the region of the neck and the sample breaks. Only one neck is developed in the sample independently of its length. In a bistable assembly, the activated waiting link arrests the development of the

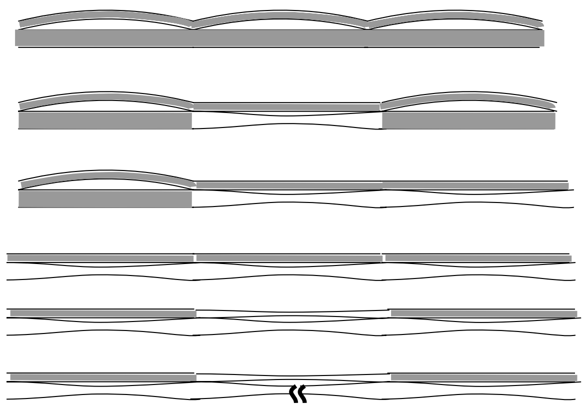

Fig. 1. Progressive damage of elastic–plastic chain under tensile loading. The grey elements are undamaged, the white elements are partially damaged.

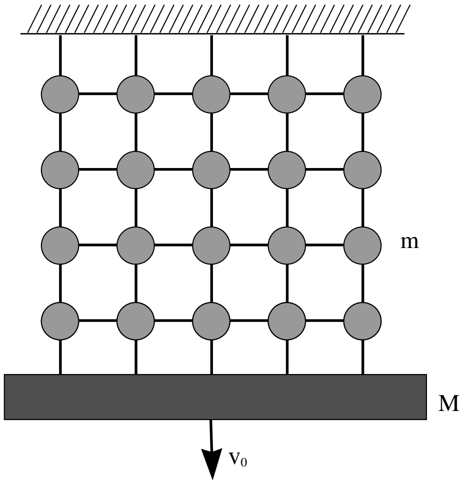

Fig. 2. Structure of the rectangular lattice.

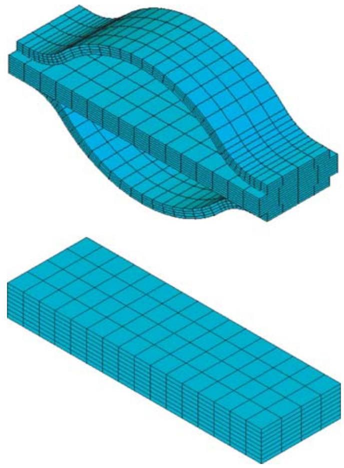

Fig. 3. Meshed models for the bistable and solid links.

irreversible plastic deformation and prevents the failure of the necked main link. When the elongation of the whole chain increases, another neck is formed in a neighbor assembly instead, and the damage propagates. This sequentially occurred transformation from one stable state to another creates a transition wave that propagates along the chain. The process continues until all links experience the transition, then the deformations in the waiting links also become plastic and the chain breaks.

Notice that the elongation of a rectangular lattice under plain normal impact is equivalent to the elongation of the chain (Fig. 2). The links that are perpendicular to the loading are not stressed. They enter the equations only by the addition of their masses at the joints. The static behavior is analogous to the behavior of a chain.

Compared with the elastic–brittle lattices investigated earlier, the elastic–plastic protective structures naturally sustain larger deformations which increases the structural strength. The waiting link may be made considerably thinner in comparison to the main link, because its role is to stop the fatal instability of the necking. To the contrary, the elastic–brittle structure will survive the failure of a main link only if the waiting link is thicker than the main one.

2.2. Simulation of the bistable link

The finite element simulation is used to demonstrate the bistable force–displacement relation. Copper C26000 was chosen because of its low hardening and high ductility. Geometrically, the assembly consists of the main link (25.4 mm × 7.94 mm × 1.59 mm) in the center and two equal waiting links (27.94 mm × 6.35 mm × .79 mm) at the top and bottom, and joined at the ends. This design provides a symmetric deformation of the main link.

The simulations are carried out using ANSYS 8.0. In a simulated model of the bistable link, the geometric and material non-linearities and finite deformations are taken into account. The 3-D model is used in order to correctly simulate necking. The finite element simulation is performed using the 8-node element having three degrees of freedom at each node (SOLID45). The meshed model of the structure is shown in Fig. 3. Finer mesh resolution does not yield noticeable differences in the results. To simulate the tensile experiment, the displacement is applied in the longitudinal direction at both ends. The symmetry is used and the simulation is performed on one fourth of the structure. The following boundary conditions are imposed at the end of the link where the waiting and main links are joined: The vertical displacement is zero, the horizontal displacement is constant at each time instance.

The elastic–plastic response of the assembly is simulated using von Mises yield criterion, associated flow rule, and isotropic work hardening. The computed deformation of the structure is shown in the plot of the von Mises stress (Fig. 4) at the final load step. For better visibility, the second contour plot represents only a quarter of the structure. Notice that the neck in the main link is developed and arrested when the waiting link is straightened. Fig. 5 shows the computed force–displacement

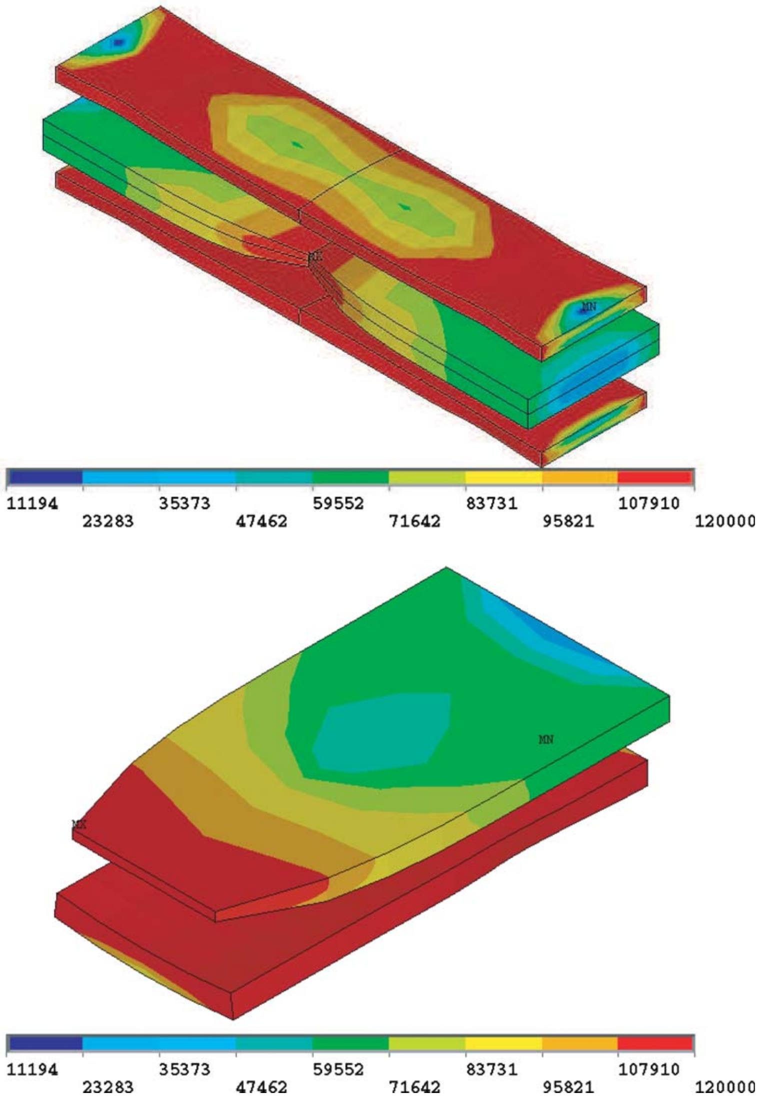

Fig. 4. Bistable link: Contour plot of von Mises stress at the final load step.

relation for the main and waiting link. The bistable force–displacement relation of the whole assembly shown in Fig. 6 is obtained by summing the resistance force in the main and waiting links. Notice that force–elongation relation does not depend on the location of the neck; in the following simulations, the neck occurs in the center of the links.

The bistable link is compared with the solid link of the same length, mass, and material, see Fig. 3. The solid link also develops a neck, see Fig. 7, but lacks the second stable interval of elongation. The force–elongation dependence of the solid link is shown at Fig. 8. In this paper, we do not perform a detailed investigation of the dependence on the thickness of the links, but restrict ourselves to several examples.

3. Dynamics of a chain

3.1. Assumptions and equations

Addressing the dynamics of bistable chains (lattices), we use a simplified model. We assume that

(1) The simulated force–elongation dependencies in the bistable and solid links are replaced by the piece-wise linear functions, see Figs. 6

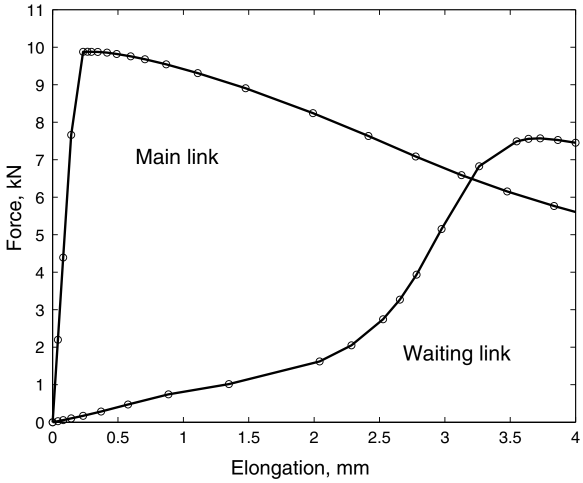

Fig. 5. Force–displacement diagram for the main and waiting links. Notice the large interval of the decreasing force in the main link due to the necking. The S-shaped diagram of the initially curved waiting link is due to its straightening.

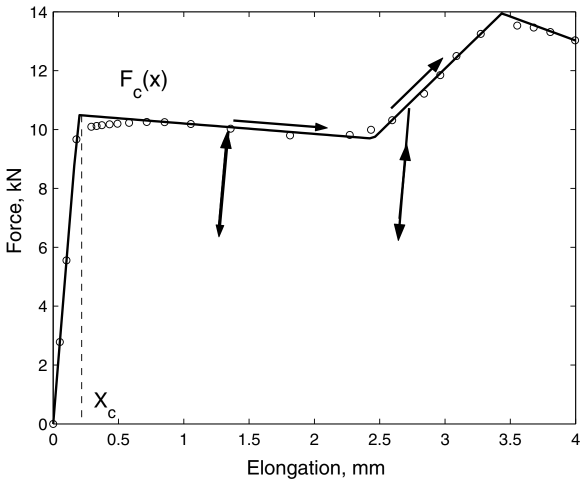

Fig. 6. Force–displacement diagram of a bistable link and its piece-wise linear approximation.

and 8. The figures show that the approximation is accurate; it allows for accelerating of the computations.

(2) The model separates the concentrated masses and massless springs. The mass of a whole link is concentrated in the joints between them. This assumption is conventional for structural dynamics. It greatly simplifies the calculation because the chain is replaced by a mass-spring system. The neglected degrees of freedom correspond to the relative motion of different parts of the link. This motion is of very high frequency and it is beyond the scope of the assumptions of elastic–plastic behavior (we do not account for viscosity), and are not in the focus of the investigation. Here, we concentrate on the propagation of the wave of transition that required the modeling for much lower frequencies. For the lattice model (Fig. 2), the additional masses of perpendicular unstressed links are added to the masses of the stressed links at the joints.

(3) The simulated force–elongation dependence corresponds to the static response of the material. It is assumed that the same dependence describes the response to a dynamic loading. The interval of the velocities of the projectile used in the calculations does not exceed 1.2% of the sound speed in the material which justifies this assumption.

(4) The unloading of the elastic–plastic link follows the straight path parallel to the linear (elastic) part of the force–elongation diagram. This assumption simplifies the model; the details of this diagram seems to be not very important for our goals.

The elastic–plastic deformation of the link is described by the equation [for detailed discussion of the plastic flow, see Hill (1998), Khan and Huang (1995)]

$$F(t,x,\dot{x}) = F_c(x_{pl}(t)+x_c) - c(x_{pl}(t)+x_c-x(t)) \tag{1}$$

$$\dot{x}_{pl}(t) = \begin{cases} 0 & \text{if } x \leqslant x_c \quad \text{or } F < F_c \\ 0 & \text{if } F = F_c, \quad \dot{x}(t) \leqslant 0 \\ \dot{x}(t) & \text{if } F = F_c, \quad \dot{x}(t) > 0 \end{cases} \tag{2}$$

where x is the elongation of the link, x_c is the elongation of the elastic limit, x_{pl} is the plastic (irreversible) elongation, and c is the elastic constant. The critical forces $F_c(x)$ for the bistable and solid links are shown in Figs. 6 and 8, respectively. They are piece-wise linear functions of x. Notice that $F_c(x) = 0$ if the elongation exceeds a threshold, which corresponds to the failure. The initial condition

$$x_{pl}(0) = 0$$

assumes that initially there is no plastic deformation.

We perform the following numerical experiment. A chain is fixed at one end and impacted by the projectile of mass M at the other end. After the impact,

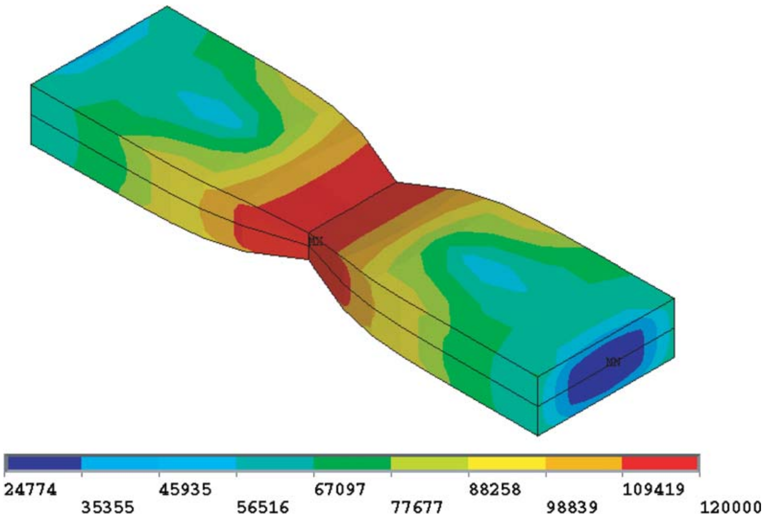

Fig. 7. Solid link: Contour plot of von Mises stress at the final load step.

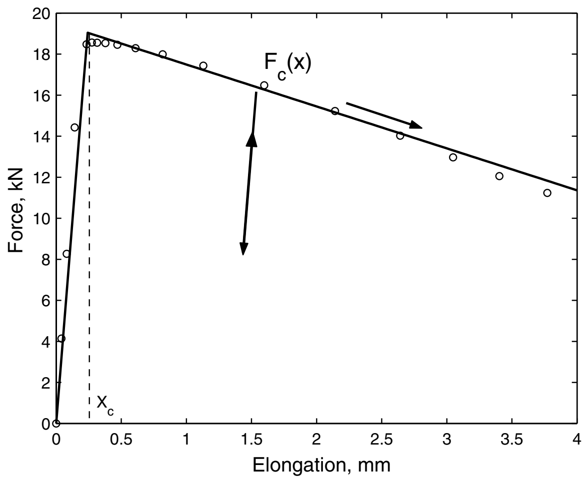

Fig. 8. Force–displacement diagram in the solid link and its piece-wise linear approximation.

the mass M stays attached to the free end of the chain. The dynamics of the chain is described by the set of differential equations:

$$m\ddot{z}_n = F(t, x_n, \dot{x}_n) - F(t, x_{n-1}, \dot{x}_{n-1}), \tag{3}$$

$$x_n = z_{n+1} - z_n, \quad n = 1, \ldots, N - 1 \tag{4}$$

$$(M + m)\ddot{z}_N = -F(t, x_{N-1}, \dot{x}_{N-1}) \quad z_0 = 0, \tag{5}$$

where z_n is the position of nth knot, x_n is the elongation of nth link, N is the number of links in the chain, and z_0 is the position of the fixed end of the chain. Mass m of the knot is equal to the mass of the link. In the lattice model, the mass m is equal to the double mass of the link, so that the masses of the unstressed links in the lattice (see Fig. 2) are taken into accounted. This system is integrated with the initial conditions

$$z_n = na \quad \dot{z}_n = 0, \quad n = 1, \ldots, N - 1, \tag{6}$$

$$z_N = Na, \quad \dot{z}_N = \frac{M}{M + m} v_0, \tag{7}$$

where a is the length of the unstressed link, and v_0 is the speed of the impact. This impact delivers the impulse Mv_0 to the last mass in the chain. The impulse corresponds to the initial speed $\frac{M}{M+m} v_0$ of that mass m and the attached mass M of the projectile.

4. Simulation and results

4.1. Waves of transition

The system of differential equations is solved using MATLAB. The typical picture of the motion of the masses in a bistable chain is shown in Fig. 9. The impact initiates an elastic wave which propagates along the chain. When the wave reflects from the support, its amplitude doubles. If the magnitude of the wave is large enough, it transfers the first link into the plastic unstable state. After this, the wave of the transition is initiated and propagates toward the impacted end. Simultaneously, the impact originates another transitional wave which propagates toward the support. The chain breaks when these two waves meet. If the speed of the impact is larger, the chain fails immediately after the impact when

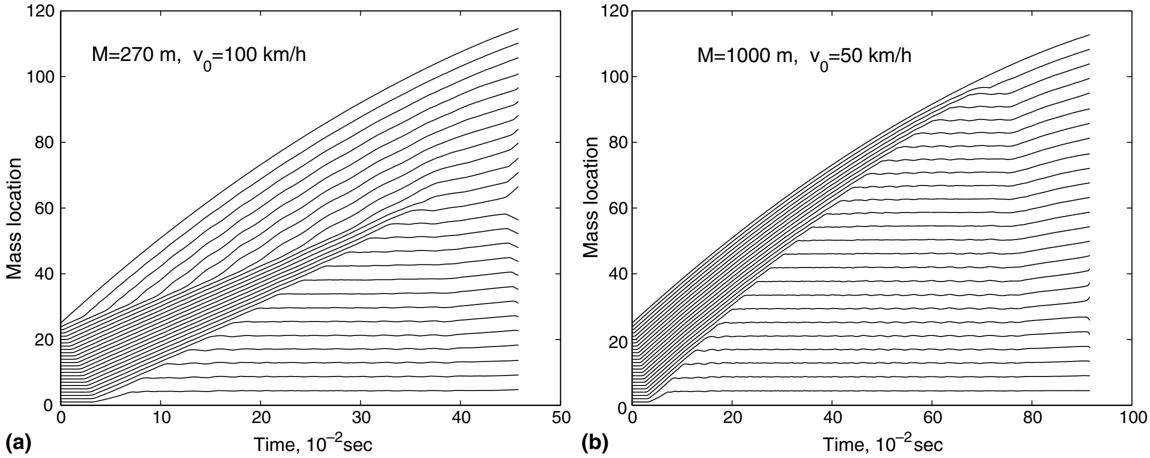

Fig. 9. Waves in elastic–plastic chain of 25 bistable links. (a) Observe the fast sonic wave, two waves of transition (damage) and the breakage when they meet. (b) The fast sonic wave initiates one transition wave.

the first link breaks, as shown in Fig. 10(a). If the speed of the impact is smaller, the transition waves do not break the chain when they meet. The even smaller speed corresponds to the absence of one of the transitional waves (see Fig. 9(b)) or both of them.

This behavior should be compared to the waves in a chain of the conventional links where no transitional waves occur, see Fig. 10. The elastic wave from the impact propagates along the chain. When it reflects from the support, it transfers the first link into the plastic unstable state and breaks it, as in Fig. 10(b). The impact of a larger speed breaks the link closest to the impact, as in Fig. 10(a). If the magnitude of the wave is smaller, the chain stays unbroken.

4.2. Energy absorption

Let us compare the state of the links in the broken conventional and bistable chains. All the links of the broken solid chain are undamaged but one. Only the damaged link experiences the plastic deformation and absorbs the energy. Contrary to this, many or all the links in the bistable chain experience a partial damage before breakage. Each of the partially damaged link is plastically deformed and absorbs the energy. Therefore, the bistable chain absorbs larger energy before the breakage. In the optimally designed chain all the links are partially damaged and absorb the energy. The bistability leads to an unexpected result: *the maximal absorbed energy in the bistable chain is proportional to its*

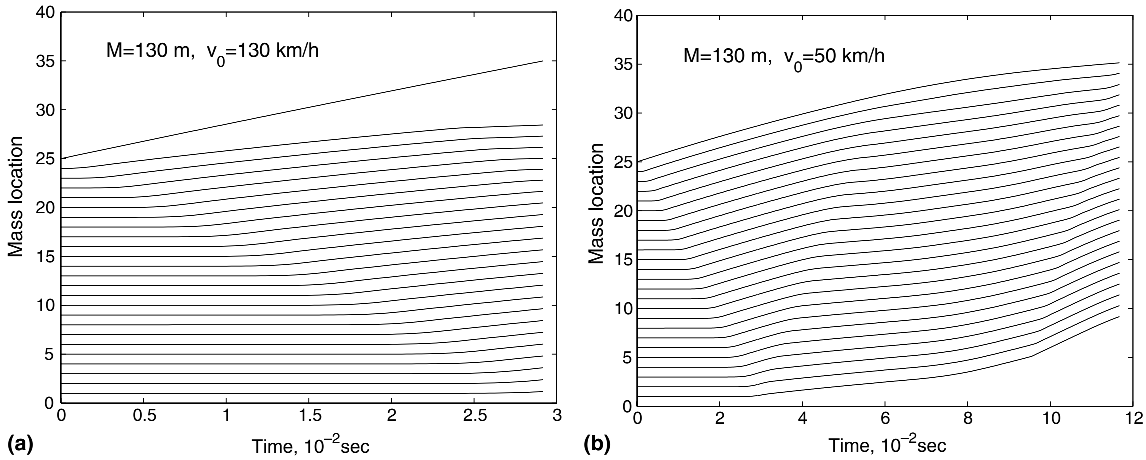

Fig. 10. Waves in elastic–plastic chain of 25 solid links. The first or the last links fail.

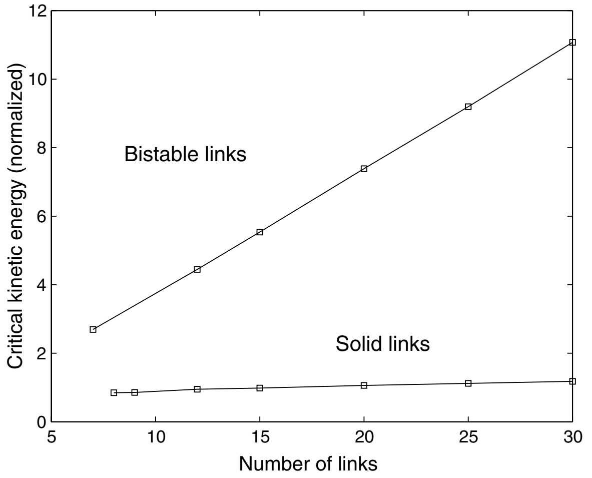

Fig. 11. Kinetic energy needed to break the bistable and conventional chains versus the number of links.

volume, *not to the cross-section* as in the conventional chain, as illustrated in Fig. 11.

Fig. 12 compares the resistance of the bistable and conventional chains. The picture shows the kinetic energy of the projectile that breaks the chain. The energy is computed at the instance of the breakage and it is zero if the projectile is stopped. The bistable chain of the same mass and length is able to capture a faster projectile. Even if it is broken, the projectile has lower residual speed. When the speed of the projectile is very large, both the bistable and conventional chains are broken by the projectile of the same energy; in that range, the link closest to the impact is broken and the other links do not contribute.

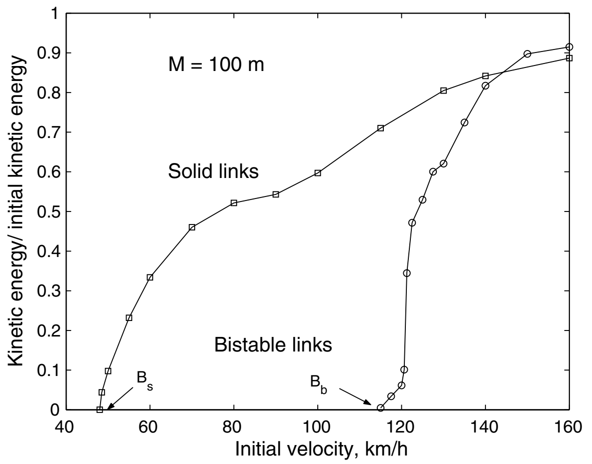

Fig. 12. Ratio of the kinetic energies of the projectile before and after collision with the chain of 15 links. Zero corresponds to captured projectile.

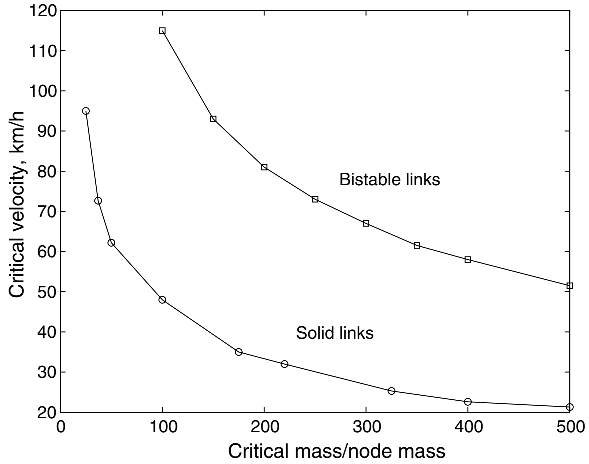

Fig. 13. The mass versus velocity of the projectile that breaks the chain.

4.3. Critical loading

The points B_s and B_b shown in Fig. 12 correspond to the critical behavior of the projectile: It breaks the chain but its residual speed is reduced to almost. If the speed of the projectile is smaller, it reverses its direction and starts to oscillate. We call a projectile *critical* if it is stopped by the chain while any projectile with larger mass or velocity breaks it. The velocity of the critical projectile vanishes at the moment when the chain fails and its elongation reaches the maximum. The critical projectile is described by two parameters, mass M and velocity v_0. These critical parameters are important characteristics of the chain because they determine its capability to resist

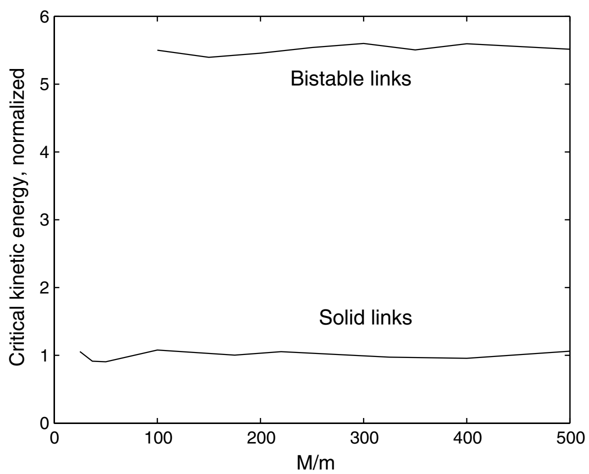

Fig. 14. Kinetic energy of the projectile that breaks the chain.

the impact. Fig. 13 compares the critical parameters of the bistable and conventional chains of the same mass, length, and material. Notice that the kinetic energy of the critical projectile stays approximately constant. The simulated bistable chain of 15 links shows *more than five times higher* critical energy than the conventional chain (see Fig. 14).

5. Conclusions

- The bistable structures can absorb many times more energy than the conventional structures of the same size, mass, and material before failure due to an impact. The amount of the energy needed to break the chain is proportional to the volume of the chain not its cross-section.
- Damage is not accumulated in a certain location, but is uniformly distributed over the structure. The bistable chains are capable of spreading the damage along the chain.
- The superior impact resistance is achieved by the design of the links that leads to bistability of the effective force–elongation relation. The bistable structures experience the phase transition that is accompanied by intensive energy absorption.

Acknowledgments

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