$8^{\text{th}}$ **World Congress on Structural and Multidisciplinary Optimization**
June 1-5, 2009, Lisbon, Portugal

# Optimal structures of multiphase elastic composites

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## 1. Abstract
We find plane periodic three-material composite structures of maximal stiffness, which is the composites of minimal elastic energy in a given homogeneous anisotropic stress field. One of the materials is assumed to be very weak (void), and the two others are linearly elastic and isotropic. A similar problem for two-material mixtures was solved 25 years ago (Gibiansky and Cherkaev, 1984) and it was shown that the second-rank laminates are optimal because they correspond to translation bound for the energy. Since then, the theory of bounds for the two-material composite was developed and other types of optimal structures were found. The generalization of the results to multimaterial case is nontrivial and requires new ideas. Here, the new bounds are established for the energy of a periodic cell and new types of microstructures are suggested that either exactly realize these bounds, or approximate them. We find these bounds using localized polyconvexity method. The bounds are geometrically independent: they depend only on elastic moduli of the materials, their volume fractions, and the anisotropy of a homogeneous external loading. The found optimal structures vary with the loading anisotropy degree. We show that there are several topologically different structures and several algebraically different bounds that are optimal in different parameter domains. All the microstructures are found by the same procedure based on (i) the energy bounds and sufficient optimality conditions for stress fields inside each material, and (ii) the lamination technique that allows for satisfaction of these conditions.
**2. Keywords:** multimaterial composites, optimal microstructures, structural optimization, bounds for effective properties,

## 3. Introduction
We consider the problem of optimal micro-geometries of multimaterial elastic composites (plane problem), aiming to maximize the stiffness of the composite in a given homogeneous stress field. Several types of optimal two-material micro-geometries are described in such papers as [10, 9, 12, 2, 16, 7]. Optimal structures depend on the degree of anisotropy of the stress loading. Their topology is simple and intuitively clear: for moderately anisotropic loading, the stronger material "wraps" the weaker one so that the weak material forms an nucleus, and the strong one - a core. The structure adjusts itself to meet the *sufficient* optimality conditions, which are found independently from solving the problem of geometrically independent bounds for effective properties. They state that the stress field in nucleus is isotropic and the sum of absolute values of the main stresses in the core is constant. For very anisotroptic loading, the structure degenerates into laminates.The results are summarized in books [4, 2, 3].

The problem of optimal three-material composite is far more complex and optimal structures are more diverse. This time, the optimal topology depends on volume fractions of the mixing elements. The general theory of multiphase exact bounds and structures is not worked up yet, although many partial results are obtained, see [14, 11, 15, 8, 6]. We consider here the simplest problem of this kind, assuming that two materials have zero Poisson coefficients and the third one is void, and still obtain a variety of optimal geometries. We exploit the method of *localized polyconvexity* developed previously in [5] for solution of similar problem for bounds of isotropic composites. The method is based on the procedure by Nesi [13] that combines the translation method [4] and additional inequality constraints. We extend the bounds [13, 5] to anisotropic composites. The obtained bounds and corresponding *sufficient conditions* for the stress in the materials in optimal structures are then used to determine optimal laminate structures, as in [5]. Depending on anisotropy of the loading and volume fractions, we find several types of structures using the technique of *inverse laminating*, see [1, 5]. In all cases but one, the found structures achieve the bounds. In the remaining case, the found structures approximate the bound.

The considered problem is an essential part of a more general problem of optimal multimaterial design. In the general problem, one asks about an optimal layout of materials in a domain that is subject to a fixed boundary traction. It turns out that the optimal layout is highly heterogeneous and there are domains where the materials mix in an infinitesimal scale to achieve the optimality. The microstructures of these optimal mixtures are investigated here. The found optimal structures are distributed in a large scale in an optimal design, according to the stress field in it.

## 4. The problem

Assume that three materials are mixed forming a periodic composite. The materials occupy plain domains $\Omega_i,\ i = 1,2,3 \subset R_2$ that form a unit periodicity cell,

$$\bigcup_{i=1,2,3} \Omega_i = \Omega$$

where $\Omega$ is a unit square, $\Omega = \{(x_1, x_2):\ 0 \le x_1 < 1,\ 0 \le x_2 < 1\}$. The areas $m_i = \|\Omega_i\|$ of $\Omega_i$ are fixed:

$$m_1 + m_2 + m_3 = 1, \quad m_i \ge 0, \tag{1}$$

otherwise, $\Omega_i$ are arbitrary sets.

The periodic composite is subject to an arbitrary homogeneous stress field $\sigma_0$ applied at infinitely distant points. The stress tensor

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

satisfies the constraints (equilibrium conditions)

$$\sigma = \sigma^T,\ \forall x \in \Omega, \tag{2}$$

$$\begin{array}{l} \frac{\partial}{\partial x_1}\sigma_{11} + \frac{\partial}{\partial x_2}\sigma_{12} = 0 \\ \frac{\partial}{\partial x_1}\sigma_{21} + \frac{\partial}{\partial x_2}\sigma_{22} = 0 \end{array} \quad \forall x \in \Omega \tag{3}$$

in each point of the domain, and the integral constraint

$$\int_\Omega \sigma\, dx = \sigma_0. \tag{4}$$

We consider the simplest problem of optimal multimaterial composites: Assume that one of the material (material No 3) is void, its compliance tensor is infinite, stress is zero

$$\sigma = 0 \quad \text{in } \Omega_3$$

and the strain tensor is not defined. The energy of void is presented as

$$W_3(\sigma) = \begin{cases} 0 & \text{if } \sigma = 0 \\ +\infty & \text{if } \sigma \ne 0 \end{cases}. \tag{5}$$

Assume also that the other two materials possess zero Poisson coefficients, their compliance tensors are proportional to the unit fourth-rank tensor, so that the stress is proportional to the strain. The energy of such materials has a form

$$W_i(\sigma) = \frac{1}{2}k_i Tr(\sigma^2) = \frac{1}{2}k_i(S^2 + D^2), \quad i = 1,2. \tag{6}$$

where $k_1$ and $k_2$ are the compliances of the corresponding material,

$$k_1 < k_2, \tag{7}$$

and $S$ and $D$ are the half-sum and half-difference of the eigenvalues $\sigma_\alpha$ and $\sigma_\beta$ of the stress tensor $\sigma$,

$$S = \frac{1}{2}(\sigma_\alpha + \sigma_\beta), \quad D = \frac{1}{2}(\sigma_\alpha - \sigma_\beta). \tag{8}$$

We also introduce the related spherical $s$ and deviatoric $d$ parts of $\sigma$,

$$s = \frac{1}{2}Tr(\sigma)\mathbf{I}, \quad d = \sigma - s, \quad (Tr\, d = 0). \tag{9}$$

Because Poisson coefficient is zero, the problem is invariant to the change of sign of the eigenvalues $\sigma_\alpha$ and $\sigma_\beta$, which allows of considerating only the case

$$\sigma_\alpha \geq 0 \quad \sigma_\beta \geq 0 \tag{10}$$

and significantly simplifies the notations.

The energy of the cell has the form

$$En(\sigma_0, k_i, \Omega_i) = \min_{\sigma \text{ as in } (2),(3),(4)} \sum_{i=1}^{2} \int_{\Omega_i} W_i(\sigma) dx. \tag{11}$$

We find a lower geometrically independent bound for the energy by arbitrary varying subdomains $\Omega_1$ and $\Omega_2$ while preserving their areas (fractions of the materials in the composite)

$$B(\sigma_0, k_i, m_i) = inf_{\Omega_i : |\Omega_i| = m_i} En(\sigma_0, k_i, \Omega_i). \tag{12}$$

The bounds for the stiffness (the energy) quadratically depend on eigenvalues of $\sigma_0$ and have the form

$$B(\sigma_0, k_i, m_i) = \sigma_{0\alpha}^2 W(k_i, m_i, r), \tag{13}$$

where

$$r = \left| \frac{\sigma_{0\beta}}{\sigma_{0\alpha}} \right|, \quad r \in [0, 1]$$

is the ratio of eigenvalues $\sigma_{0\alpha}$ and $\sigma_{0\beta}$ of $\sigma_0$ and it is assumed that $|\sigma_{0\alpha}| \geq |\sigma_{0\beta}|$.

## 5. Technique: Localized polyconvex envelope

The technique of the derivation of the bound is called *localized polyconvexification* and is described in [5]. In the procedure, the differential constraints (3) are relaxed and replaced by an integral constraint of quasiaffiness, see for example [4]

$$\int_\Omega \det(\sigma) dx = \det(\sigma_0) \quad \text{or} \quad \int_\Omega (S^2 - D^2) dx = S_0^2 - D_0^2 \tag{14}$$

and inequalities (see Nesi inequalities [13])

$$\det(\sigma) \geq 0, \quad \forall x \in \Omega, \quad \text{if } \det(\sigma_0) \geq 0. \tag{15}$$

Notice that the procedure gives Translation Bounds without these inequalities, or when the inequalities are slack. In the following analysis (see Section 6), this case corresponds to regimes D and E. Notice that the translation bound is optimal when fraction $m_1$ is large enough.

The relaxed problem defines the bound $W$ of the energy $En$,

$$W(\sigma) \leq En(\sigma) \quad \forall \sigma \tag{16}$$

in the form

$$W = \max_{t \in R} \left( \min_{S_1, S_2, D_1, D_2 \in \mathcal{P}} (V_1' + V_2' + V_1'' + V_2'') - t \left( S_0^2 - D_0^2 \right) \right), \tag{17}$$

where

$$\begin{aligned}
V_i' &= \min_{s(x) \in \mathcal{Q}} \int_{\Omega_i} (k_i + t) \|s(x)\|^2 dx, && (18) \\
V_i'' &= \min_{d(x) \in \mathcal{Q}} \int_{\Omega_i} (k_i - t) \|d(x)\|^2 dx, && (19) \\
\mathcal{P} &= \{S_1, S_2, D_2, D_2 : \ m_1 S_1 + m_2 S_2 = S_0 = \sigma_{0\alpha}(1 + r), \\
& \quad m_1 D_1 + m_2 D_2 = D_0 = \sigma_{0\alpha}(1 - r)\}, && (20) \\
\mathcal{Q} &= \big\{ s(x), d(x) : \ \|s\|^2 \geq \|d\|^2 \ \forall x \in \Omega && (21) \\
& \quad \int_{\Omega_i} S(x)\, dx = m_i S_i, \ \int_{\Omega_i} D(x)\, dx = m_i D_i, \ i = 1, 2 \big\}. && (22)
\end{aligned}$$

Analyzing this problem, we notice several types of minimizers – stress components $s(x)$, $d(x)$ in subdomains $\Omega_i$. These minimizers corresponds to the inequality Eq.(16) and they safisfy *sufficient optimality conditions*. These minimizers-stresses are generally not compatible, and the bound is not achievable. Ineq. (21) might be either active or not. Let us comment on these conditions.

– Because of inequality Eq.(21), optimal value of $t$ is nonnegative, $t \geq 0$.

– The eigenvectors of $\sigma(x)$ are codirected with the eigenvectors of $\sigma_0$ everywhere in $\Omega$.

– Minimum in the Eq.(18) is achieved at a constant solution, $s(x) = S_i \mathbf{I}\ \forall x \in \Omega_i$.

– When $t_{opt} \in [0, k_1)$, minimum in the Eq.(19) is acheved at a constant solution, $d(x)$ =constant, $\|d(x)\| = D_i\ \forall x \in \Omega_i$ as well.

– When $t_{opt} = k_1$, the deviator $d(x)$ in $\Omega_1$ is not defined because the coefficient $k_1 - t$ becomes zero, see Eq.(19). $d(x)$ can vary without affecting the bound, if only Ineq.(21) is satisfied. In $\Omega_2$, the optimal deviator $d(x)$ is zero.

– When $t_{opt} > k_1$, the functional $V_1''$ is a concave function of $d(x)$. Its minimum corresponds to $d(x)$ that alters between the extremal values $\pm\|s\|$, see Ineq.(21). It also can stay constant and equals to either $\|s\|$ or $-\|s\|$. The equality holds $\|d(x)\|^2 = \|s(x)\|^2\ \forall x \in \Omega_1$, and the mean values of $S$ and $D$ satisfy inequality

$$-S_1 \leq D_1 \leq S_1, \quad D_1 = \frac{1}{m_1} \int_{\Omega_1} D(x)\, dx. \tag{23}$$

This inequality becomes active (satisfied as an equality) when $D(x), x \in \Omega_1$ is constant.

– When $t_{opt} < k_2$, the $D$-field in $\Omega_2$ is constant. When $t_{opt} = k_2$, the $D$-field in $\Omega_2$ is not uniquely defined, see Eq.(19).

## 6. Results: Bounds

The optimal bound for the energy is a multifaced surface: It is expessed by different analytic expressions in different domains. The results are conveniently presented in the parameters plane $r, m_1$, Figure 1. The dependence on $m_2$ is not shown on the figures. We notice that this dependence leads to transformation of the shape of the regions below but does not change their topology. We also notice that bounds depend on absolute values of the stress tensors. Therefore we assume that $\sigma_\alpha > 0$ and $\sigma_\beta > 0$. The bounds are as follows

Table 1: Bounds for energy

| | Ineq(23) in $\Omega_1$ | $t_{opt}$ | $D(x)$ in $\Omega_1$ | $D(x)$ in $\Omega_2$ | Exact? |
|---|---|---|---|---|---|
| A | slack | $k_2$ | $D = \pm S$ | $\in [-S_2, S_2]$ | yes |
| B | slack | $\in (k_1, k_2)$ | $D = \pm S$ | 0 | yes |
| C | active | $\in (0, k_2)$ | $D = S$ | cnst | yes |
| D | slack | $k_1$ | $\in [-S_1, S_1]$ | 0 | yes |
| E | slack | $\in (0, k_1)$ | cnst, $\lvert D \rvert < S$ | cnst | no |

where regions (see Figure 1) are defined as

$$\Phi_{A1} \quad : \quad \frac{(k_1 - k_1 m_2 - m_1 k_2)\, m_2}{m_1 k_2} \leq r \leq \frac{m_2 k_1^2}{\left[(m_2 k_1 + m_1 k_2) + \sqrt{(m_2 k_1 + m_1 k_2)^2 - m_2 k_1^2}\right]^2}$$

$$m_2 \leq r \leq 1, \quad \frac{k_1\left(\sqrt{m_2} - m_2\right)}{k_2} \leq m_1 \leq \frac{k_1 (1 - m_2)}{2 k_2}, \tag{24}$$

$$\Phi_{A2} \quad : \quad \frac{(k_1 - k_1 m_2 - m_1 k_2)\, m_2}{m_1 k_2} < r, \quad \text{and} \quad 0 < r \leq 1, \quad 0 < m_1 \leq \frac{k_1 (1 - m_2)}{k_2} \tag{25}$$

$$\Phi_B \quad : \quad \frac{m_2 k_1^2}{\left[(m_2 k_1 + m_1 k_2) + \sqrt{(m_2 k_1 + m_1 k_2)^2 - m_2 k_1^2}\right]^2} < r \leq \frac{4 m_2 k_1^2}{\left(a + \sqrt{a^2 - 4 m_2 k_1^2}\right)^2}$$

$$\text{where} \quad a = m_1 k_2 + m_1 k_1 + 2 m_2 k_1, \quad \text{and} \quad m_2 \leq r \leq 1 \tag{26}$$

$$\Phi_C \quad : \quad \frac{(k_1 - k_1 m_2 - m_1 k_2)\, m_2}{m_1 k_2} \leq r \leq \frac{m_2 \sqrt{k_2 (k_1 - k_1 m_2 - m_1 k_2)(1 - m_1 - m_2)}}{m_1 k_2},$$

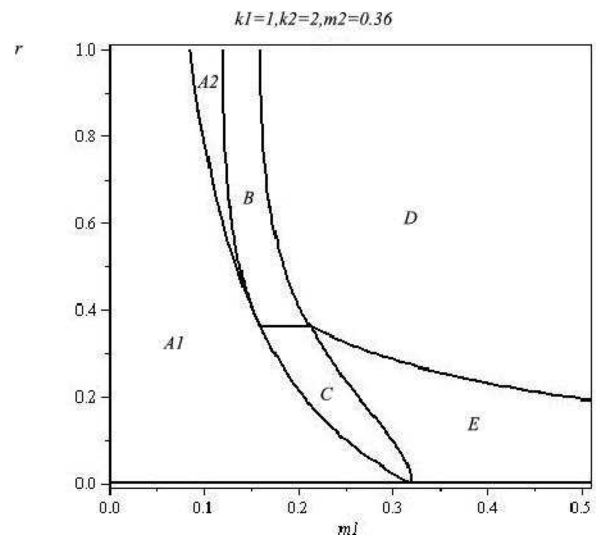


Figure 1: **Regions of multifaced boundary for the energy in an anisotropic field.**

$$0 < r \le m_2 \tag{27}$$

$$\Phi_D \quad : \quad \frac{4m_2k_1^2}{\left(a+\sqrt{a^2-4m_2k_1^2}\right)^2} < r, \quad \frac{m_2k_1}{m_1k_1+m_1k_2+m_2k_1} \le r$$

$$\text{and} \quad r \le 1, \quad m_1 \le 1-m_2 \tag{28}$$

$$\Phi_E \quad : \quad \frac{m_2\sqrt{k_2\left(k_1-k_1m_2-m_1k_2\right)\left(1-m_1-m_2\right)}}{m_1k_2} < r < \frac{m_2k_1}{m_1k_1+m_1k_2+m_2k_1},$$

$$m_1 \le 1-m_2 \tag{29}$$

The lower bound of the energy is expressed as a function of invariants $S_0, D_0$ of the external stress and the problem's parameters. The analytic expressions of the bound in the regions $A-E$ are:

$$W_A = \left[\left(\frac{m_1}{2k_1}+\frac{m_2}{2k_2}\right)^{-1} - k_2\right] S_0^2 + k_2 D_0^2, \tag{30}$$

$$W_B = \frac{k_1}{2m_1}\left[S_0 - 2\sqrt{\left({S_0}^2-{D_0}^2\right)m_2}\right]^2 + k_2\left({S_0}^2-{D_0}^2\right), \tag{31}$$

$$W_C = \left[\frac{k_1\left(1-m_2\right)^2}{2m_1} + \frac{k_2m_2}{2}\right]\left(S_0+D_0\right)^2 + \frac{k_2}{2m_2}\left(S_0-D_0\right)^2, \tag{32}$$

$$W_D = \left[\left(\frac{m_1}{2k_1}+\frac{m_2}{k_2+k_1}\right)^{-1} - k_1\right] S_0^2 + k_1 D_0^2, \tag{33}$$

$$W_E = \max_{t\in[0,k_1)} \tilde{W}_E(t), \tag{34}$$

$$\tilde{W}_E(t) = \left[\left(\frac{m_1}{k_1+t}+\frac{m_2}{k_2+t}\right)^{-1} - t\right] S_0^2 + \left[\left(\frac{m_1}{k_1-t}+\frac{m_2}{k_2-t}\right)^{-1} + t\right] D_0^2.$$

Optimal values of $t$ in the regions B and C are

$$t_{opt\,B} = \frac{k_1}{m_1r}\left(\sqrt{m_2r}\,(1+r) - 2m_2r\right) - k_2, \tag{35}$$

$$t_{opt\,C} = \frac{m_2}{m_1r}\left(\left(1-m_2\right)k_1 - m_1k_2\right). \tag{36}$$

Optimal value of $t$ in region E is a root of the equation $\frac{dW_E}{dt} = 0$. The bulky solution of the resulting fourth-order equation for $t$ is not shown here. The positions of eigenvalues of stresses-minimizers are

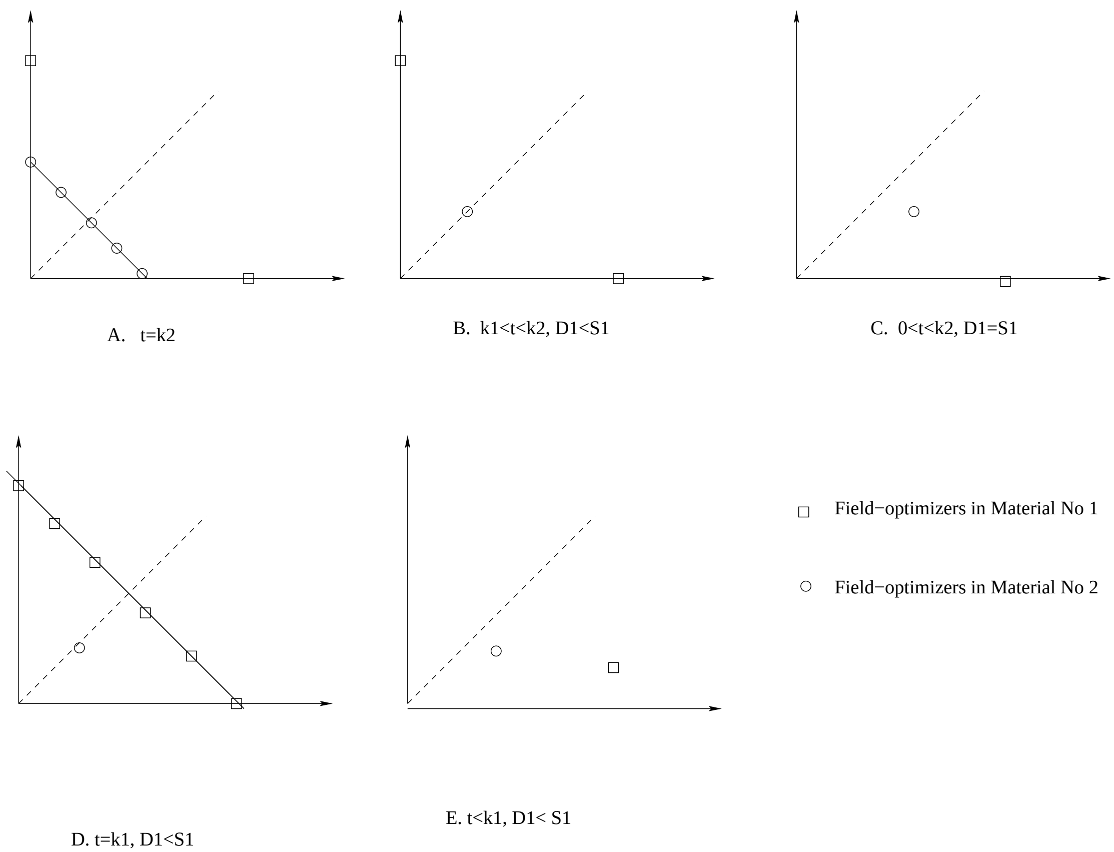


Figure 2: **The eigenvalues of the stresses-minimizers, according to the bounds. Notice that the equilibrium condition is not assumed.**

shown in Figure 2.

## 7. Results: Structures

Table 2: Optimal structures
Finally, we show the structures that realize the bounds A-D and approximate the bound E.

| Region | A1 | A2 | B | C | D | E |
|---|---|---|---|---|---|---|
| Type | (T, 2) | $(T^2, 2)$ | $(T^2)$ | (T) | $(T^2, 1, 1)$ | (T,1) |
| Optimal? | yes | yes | yes | yes | yes | unknown |

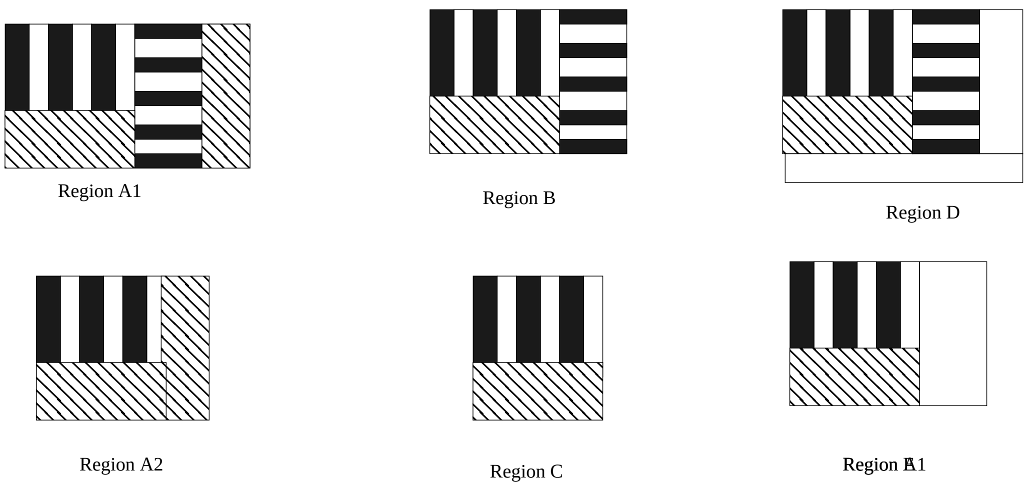


Figure 3: **The laminates. The equilibrium condition is enforced.**

**Optimal structures in regions A-D**
The cartoon of the found optimal structures is presented in Figure 3. They are sequential laminates, in other words, they are obtained by several steps, each step requires laminating of laminates obtained in previous steps.

- In region C, a simple $(T)$-structure is optimal. It is as follows: a laminate from materials 1 and 3 is laminated in an orthogonal direction with a layer of material 2. The stress fields in the materials are constant and rank-one connected, i.e. $\det(\sigma_i - \sigma_j) = 0$. We can show that the $(T)$-structure is optimal by a direct energy calculation and by noticing that the stresses in the structure coincide with the stresses-minimizers in Figure 2

- In region B, the $(T^2)$-structure is optimal, which is obtained from $(T)$-structure by adding a layer of laminate from materials 1 and 3, see Figure 3. The fractions are chosen so that the field in $\Omega_1$ has constant magnitude everywhere. The stresses in neighboring laminates are rank-one connected. The direct computation shows that the stresses in the optimal $(T^2)$-structure lie in the points of the stresses-minimizers, Figure 2, which are found from sufficient optimality conditions. We see that the $(T^2)$-structure is also optimal.

- The structures $(T, 2)$ and $(T^2, 2)$ are optimal in region A1 and A2 respectively, they are shown in Figure 3. We prove the optimality of these structures by the same method, comparing the stresses in layers with the ones found from sufficient optimality conditions and shown in Figure 2.

- In region D, the bound coincides with the Translation bound. The optimal structures have been found previously, in [8, 1], using the same method (that was developed there). The structures have the form $(T^2, 1, 1)$ and are shown in Figure 3.

At the boundaries of the regions, the corresponding structures meet each other. For example, the $(T^2)$ structure becomes $(T)$-structure when the additional layers disappear.

**Conjectured optimal structures in region E.**
Finally, we guess optimal structures in case E. Notice that the $(T,1)$-structures are optimal at the boundaries with regimes C and D. These structures degenerate into structures found in [6] for asymptotic case approaching one-directional load. They also degenerate into laminates when $m_3 \to 0$. We conject that they stay optimal inside region E. However, these or other laminate structures cannot realize the corresponding bounds because the bounds require that stress in all three phases is constant and these fields are not rank-one connected. Likely, the bound is not exact in region $E$ and its improvement would require consideration of other complementary inequalities that are not revealed yet.

## 8. Stress in optimal laminates

Here we will give an example of how to find the optimal structure. The structure discussed here is optimal in region $\Phi_{A_2}$. It corresponds to translation parameter $t = k_2$ and is a second rank laminate. It is formed by first laminating material 1, 2 and 3 along $x_2$ direction with relative volume fraction of material 1 equaling to $\mu_{11}$, relative volume fraction of material 2 equaling to $\mu_{12}$ and then adding material 2 to the resulting laminate along $x_1$ direction with relative volume fraction of material 2 equaling to $\mu_2$. Again $x_1$ and $x_2$ are orthogonal to each other. Let vectors $s_{ij}$ represent the fields in each material with the first subscript $i$ describing the layer and second subscript $j$ describing the material considered under given external stress field $\sigma_0 = [\sigma_{01} \quad \sigma_{02}]$. The first element of $s_{ij}$ codirects with the eigenvector corresponding to $\sigma_{01}$ and the second one codirects with the eigenvector corresponding to $\sigma_{02}$. As shown in Figure 4, the field inside material 1 is constant and so is the trace of the field inside material 2. Layers of materials 1, 2 and 3 are rank-1 connected (see Figure 4), that is the stress in the normal direction of the interface between any two materials and the strain in the tangential direction are continuous. Based on this, we

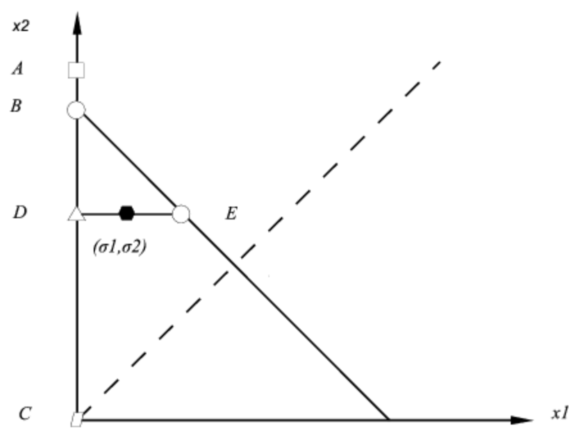


Figure 4: **The fields inside the structure T-2.**

have the following:

$$
\begin{aligned}
s_{11} &= [0, \quad \alpha], && (37)\\
s_{12} &= \left[0, \quad \frac{k_1 \alpha}{k_2}\right], && (38)\\
s_{22} &= [\beta, \quad \sigma_{02}]. && (39)
\end{aligned}
$$

The average field (represented as point D in Figure 4) in the first layer (material 1, 2 and 3) is:

$$
s_{10} = s_{11}\mu_{11} + s_{12}\mu_{12} = \left[0, \quad \alpha\mu_{11} + \frac{k_1\alpha}{k_2}\mu_{12}\right] \qquad (40)
$$

and it is rank-1 connected with the field (see point E in Figure 4) inside the second layer of material 2, therefore the following is true:

$$\alpha\mu_{11} + \frac{k_1\alpha}{k_2}\mu_{12} \quad = \quad \sigma_{02}. \tag{41}$$

Also the average field between second layer of material 2 (point E in Figure 4 ) and average field of the first rank laminate (material 1, 2, and 3) equals to the external field, and this leads to:

$$\beta\mu_2 \quad = \quad \sigma_{01}. \tag{42}$$

The fact that the trace of fields inside material 2 is constant requires that:

$$\beta + \sigma_{02} \quad = \quad \frac{k_1\alpha}{k_2}. \tag{43}$$

The restrictions on volume fractions are of the following:

$$\begin{aligned} \mu_{11}\left(1-\mu_2\right) &= m_1, & (44)\\ \mu_{12}\left(1-\mu_2\right)+\mu_2 &= m_2. & (45) \end{aligned}$$

Solve (41)-(45), we obtain:

$$\begin{aligned}
\alpha &= \frac{k_2\left(\sigma_{01}+\sigma_{02}\right)}{m_1k_2+k_1m_2}, & (46)\\
\beta &= \frac{k_1\left(\sigma_{01}+\sigma_{02}\right)}{m_1k_2+k_1m_2}, & (47)\\
\mu_{12} &= \frac{m_2\sigma_{02}}{\sigma_{01}+\sigma_{02}} + \frac{m_1k_2\sigma_{01}}{\left(m_1k_2+m_2k_1-k_1\right)\left(\sigma_{01}+\sigma_{02}\right)}, & (48)\\
\mu_{11} &= \frac{m_1\sigma_{02}}{\sigma_{01}+\sigma_{02}} - \frac{m_1k_1\sigma_{01}}{(m_1k_2+m_2k_1-k_1)(\sigma_{01}+\sigma_{02})}, & (49)\\
\mu_2 &= \frac{\sigma_{01}(m_1k_2+k_1m_2)}{\sigma_{01}k_1-\sigma_{02}\left(k_1-m_1k_2-m_2k_1\right)}. & (50)
\end{aligned}$$

Recall that $\sigma_{01} = \sigma_{02}r$ and substitute this condition into (48)-(50) and we get:

$$\begin{aligned}
\mu_{12} &= \frac{m_2}{1+r} + \frac{rm_1k_2}{\left(m_1k_2+m_2k_1-k_1\right)(1+r)}, & (51)\\
\mu_{11} &= \frac{m_1}{1+r} - \frac{rm_1k_1}{\left(m_1k_2+m_2k_1-k_1\right)(1+r)}, & (52)\\
\mu_2 &= \frac{r\left(m_1k_2+k_1m_2\right)}{rk_1-\left(m_1k_2+m_2k_1-k_1\right)}. & (53)
\end{aligned}$$

Requiring that all the volume fractions fall into (0, 1], we have the following restriction on the values of $r$ which correspond to the optimal structures:

$$0 < r < \frac{m_2\left(k_1-m_1k_2-k_1m_2\right)}{k_2m_1}. \tag{54}$$

Note that if $r = 0$, then $\mu_2 = 0$, which means the structure degenerates into laminate, and if $r = m_2(k_1-m_1k_2-k_1m_2)/(k_2m_1)$, then $\mu_{12} = 0$, which means that the structure degenerates into $T$-structure.

## 9. Discussion

1. One can show that the obtained results degenerate into known bounds/structures for two-material problem if any of the volume fractions vanishes, or if $k_1 = k_2$, or $k_2 \to \infty$.
2. Generalization of the obtained results to non-zero Poisson coefficients seems to be straightforward but the formulas will be more bulky. The case of three nonzero materials is more knotty, and one expects new domains of analyticity of the bound and new types of matching structures to appear.
3. The gap between the bound and the structures in Region E cannot be closed by the localized polyconvexity method because the bound corresponds to incompatible fields in materials. It is surprizing that the method allows for exact bounds in the other regions.
4. Unlike two-material case, the optimal three-material structures undergo several topological transitions. Either the first or second material or none of the materials forms a connected domain, depending on volume fractions and the degree of anisotropy.

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