A CLASS OF OPTIMAL TWO-DIMENSIONAL MULTIMATERIAL CONDUCTING LAMINATES

NATHAN ALBIN[1], ANDREJ CHERKAEV[2], AND VINCENZO NESI[3]

Abstract. We introduce a family of optimal anisotropic two-dimensional multimaterial laminate composites which correspond to extreme overall conductivity. These laminates attain the translation bounds and generalize all previously known constructions for these bounds. The method of construction is based on the analysis of the fields in optimal structures.

1. Introduction

The problem of the optimal structure of a periodic composite has been the subject of substantial work in various communities. Since the pioneering work of Hashin and Shtrikman (1962), two techniques have been used to solve the problem. On one hand, *outer bounds* on the effective tensors are established, which depend only on the physical properties of the constituent materials and on their relative volume fractions. One the other hand, the effective tensors of periodic microstructures are used to establish an *inner bound* on the set of effective tensors. An outer bound is found to be optimal if it coincides with an inner bound.

Though the G-closure problem for two conducting materials in two dimensions was solved more than twenty years ago in Hashin and Shtrikman (1962); Tartar (1979); Lurie and Cherkaev (1984); Tartar (1985), the solution for three-material mixtures is still not known. The translation bound, which is related to the polyconvex envelope of an auxiliary energy, is always attainable for two-material composites. However, for multimaterial composites the bound is attainable only in a special range of volume fractions of the components Hashin and Shtrikman (1962); Milton (1981); Lurie and Cherkaev (1985); Milton and Kohn (1988); Gibiansky and Sigmund (2000). Additionally, there are results for improved bounds in the case of small volume fractions of the best or worst conductor Talbot et al. (1995); Nesi (1995).

In this paper, we construct a family of structures which realize the translation bound by analyzing the (pointwise) fields in optimal structures. In

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particular, our family of structures generalizes the structures of multimaterial composites found in Milton and Kohn (1988), and those found in Gibiansky and Sigmund (2000). Additionally, we discuss a new pointwise constraint on the fields in the materials inside any translation-optimal structure which supplements the translation bound. This constraint determines a new necessary condition for the attainability of the translation bound. Our method is based on the analysis of the fields in optimal structures. The results are presented for *two-dimensional linear conductivity*, although much of the method applies to various other types of physical phenomena both in two and three dimensions.

2. Notations and bounds

2.1. **Multiphase conducting mixtures.**

Consider a two-dimensional periodic multiphase structure. The unit periodicity cell $\Omega = [0,1]^2$ is divided into N parts $\Omega_1, \dots, \Omega_N$ occupied with materials with isotropic conductivity tensors

$$K_i = k_i I \quad \text{for } i = 1, ..., N \tag{1}$$

where I is the two-by-two identity matrix. We assume the conductivities are ordered so that $0 < k_1 < \dots < k_N$. The conductivity equations applied to the periodicity cell are written as

$$\operatorname{div} K(x) \nabla u(x) = 0 \text{ in } \Omega, \quad \int_\Omega \nabla u(x)\, dx = e \tag{2}$$

where $K : \Omega \to \{K_1, ..., K_N\}$ is the variable conductivity tensor defined by

$$K(x) = K_i \quad \text{if } x \in \Omega_i, \quad i = 1, \dots, N, \tag{3}$$

$K_1, ..., K_N$ are given by (1), and where e is the prescribed external field acting on Ω.

Assume that the periodicity cell with material layout defined by $K(x)$ is subject to the homogeneous external field e that is gradient of a linear potential $e^T x$. The energy stored in the material is given by

$$W(K,e) = \inf_{u \in H^1_\#(\Omega) + e\cdot x} \int_\Omega \nabla u(x) \cdot K(x) \nabla u(x)\, dx$$

where $H^1_\#(\Omega)$ is the space of locally H^1 functions on $\mathbb{R}^2$ which are Ω-periodic with zero mean. The infimum is taken over functions with fixed affine part plus a variable periodic oscillating part:

$$u(x) = e \cdot x + \operatorname{osc}(x), \quad \int_\Omega \nabla u(x)\, dx = e.$$

The affine part, $e \cdot x$, is prescribed by the loading. The minimization is taken over the zero-mean oscillating part, $\operatorname{osc}(x)$.

The effective tensor K^* of the structure with the partition Ω_i is defined as a homogeneous material that stores the same energy as the mixture under the same homogeneous loading. That is,

$$e \cdot K^* e = \inf_{u \in H^1_{\#}(\Omega) + e\cdot x} \int_\Omega \nabla u(x) \cdot K(x) \nabla u(x)\ dx \quad \forall e \in \mathbb{R}^2.$$

In order to completely determine K^*, it suffices to consider the response of the same structure to the two orthogonal loadings, see Lurie and Cherkaev (1984); Francfort and Milton (1994).

$$e = e_1 = r_1(1,0)^T \quad \text{and} \quad e = e_2 = r_2(0,1)^T. \tag{4}$$

The response in this case means the sum of the energies of these loadings:

$$W(K, e_1) + W(K, e_2). \tag{5}$$

This functional can be conveniently rewritten in terms of two-by-two matrices. We write (4) as

$$E = \operatorname{diag}(r_1, r_2). \tag{6}$$

Given any pair of potentials $U = (u_1, u_2)$, we define the two-by-two gradient matrix as the matrix whose rows consist of the gradients of u_1 and u_2:

$$DU = \{DU_{ij}\}_{i,j\in\{1,2\}}, \quad DU_{ij} = \frac{\partial u_i}{\partial x_j}.$$

The sum of energies (5) becomes

$$\mathcal{W}(K, E) = \inf_{U \in H^1_{\#}(\Omega)^2 + Ex} \int_\Omega \langle DU(x)\, K(x), DU(x)\rangle\ dx$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined on two-by-two matrices by

$$\langle A, B\rangle = \operatorname{tr}(AB^T).$$

The effective tensor K^* is the unique (symmetric) tensor satisfying the relation

$$\langle E\, K^*, E\rangle = \inf_{U \in H^1_{\#}(\Omega)^2 + Ex} \int_\Omega \langle DU(x)\, K(x), DU(x)\rangle\ dx \quad \forall E \in \mathbb{R}^{2\times 2}.$$

3. Bounds

3.1. **Wiener and translation bounds.** The effective conductivity satisfies the following inequality bounds, (see, for example, Cherkaev (2000); Milton (2002)).

(1) The Wiener bounds:

$$\left(\sum_{i=1}^N \frac{m_i}{k_i}\right)^{-1} \le \lambda_{\min}(K^*) \le \lambda_{\max}(K^*) \le \sum_{i=1}^N m_i k_i. \tag{7}$$

where $\lambda_{\min}(K^*)$ and $\lambda_{\max}(K^*)$ are the minimum and maximum eigenvalues of K^* respectively and where $m_i = |\Omega_i|$ are the *relative volume fractions* of the materials, $m_1 + m_2 + m_3 = 1$. The inequalities place the pair of eigenvalues of any effective tensor in a rectangular box in the eigenvalue plane. The bound is sharp: The effective tensor of the anisotropic laminate satisfies both inequalities as equalities. Moreover, for multicomponent ($N \geq 3$) structures, the bound is achieved at certain intervals of the sides of the box, Cherkaev and Gibiansky (1996).

(2) The translation bounds:

$$\frac{\operatorname{tr} K^* - 2k_1}{\det K^* - k_1^2} \leq 2 \sum_{i=1}^{N} \frac{m_i}{k_i + k_1} , \tag{8}$$

$$\frac{\operatorname{tr} K^* - 2k_N}{\det K^* - k_N^2} \geq 2 \sum_{i=1}^{N} \frac{m_i}{k_N + k_i} . \tag{9}$$

These bounds are sharp for certain values of the m_i, k_i, and the degree of anisotropy of K^* as is discussed later in this paper.

3.2. **Conditions of realizability of the translation bounds.** The translation bounds (8) and (9) are not sharp for all values of the parameters m_i and k_i. Intuitively, we see this from the fact that the formulas for the bounds still depend on k_1 (respectively k_N) when $m_1 = 0$ (respectively $m_N = 0$) as was discussed in Milton and Kohn (1988). Besides, for m_1 or m_N near 0, there are better bounds Talbot et al. (1995); Nesi (1995), so the translation bounds cannot be sharp. In the rest of the paper we focus primarily on the lower bound (8). Similar constructions exist for the upper bound (9) by duality arguments.

Theorem 1 (Realizability theorem)**.** *A structure realizes the bound* (8)*, if the following conditions hold on the pointwise field DU when the structure is placed in to a properly scaled diagonal external field E in (6) (compare to Grabovsky (1996); Milton (2002).)*

(P1) $\operatorname{tr} DU = 1$ *a.e. in* Ω_1.
(P2) $DU = \frac{k_1}{k_i + k_1} I$ *a.e. in* Ω_i *for* $i = 2, ..., N$.
(P3) DU *is diagonal in* Ω_1.

In fact, the theorem is true if (P1)-(P3) hold in an approximate sense. In particular, if the piecewise constant "fields" in a sequential laminate satisfy (P1)-(P3) then the laminate is optimal. It is in this sense that we refer to the fields in laminate structures from now on.

In addition, we show that in order for a laminate structure to satisfy the bound, the field in the first material cannot be "too anisotropic"

Theorem 2. *If a laminate structure satisfies the bound* (8) *then (under the assumptions of the previous theorem) the field in the first material must* *satisfy the relation*

$$\det DU \geq \frac{k_1 k_N}{(k_N + k_1)^2} \quad in\ \Omega_1. \tag{10}$$

Indeed, this inequality easily follows from the fact that the fields in a layer of lamination must be in a rank-one connected.

The inequality (10) limits the applicability of (8); the bound cannot be satisfied by laminates that are either extremely anisotropic or that contain too small an amount of the first material. The T^2-structures described below satisfy the condition (10) as equality and therefore represent the boundary of applicability of the translation bound for laminates.

4. The optimal structures

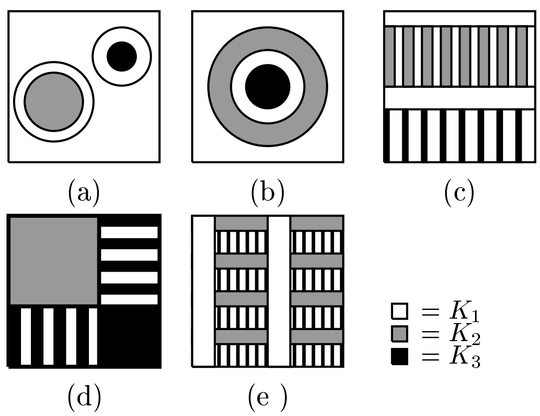

Figure 1. Previously known three-phase structures optimal for the translation bound (8).

4.1. **Known structures.** The first type of isotropic structures to attain the translation bound was described in Milton (1981) (see Figure 1a). The construction for the lower bound (8) (K^* is isotropic) is possible if m_1 is large enough. All such constructions satisfy $K_1 \leq K^* \leq K_2$. Milton's construction was extended to anisotropic composites later in Milton and Kohn (1988) (see Figure 1c). The topology of the optimal structures is not unique as follows from Lurie and Cherkaev (1985) where an alternative construction was given for structures with exactly the same volume fractions and effective properties as those presented by Milton (see Figure 1b). In Cherkaev and Gibiansky (1996) three-material anisotropic structures that have effective tensors with eigenvalues on an interval of the sides of the Wiener box not only in its corner were introduced (see Figure 1e). There are no similar structures in the two-material case.

In Gibiansky and Sigmund (2000) a new construction was described that significantly increased the set of optimal points of the translation bounds (8) and (9) for the case $N = 3$. The paper focuses on the problem of bulk moduli, but the authors extend the results to the conductivity problem as well. Their structures were the surprising result of a numerical simulation. Using a "topology optimization" algorithm developed earlier by Sigmund, the authors searched for optimal structures by computer. They found a structure which lies outside the Kohn and Milton range of parameters (m_1 is too small for any of the previous constructions to apply) but which numerically appears to satisfy the translation bound. The surprise occurred when the authors attempted to replace the computer output with a similar, but simpler structure for which the effective properties could be analytically computed. The simplified structure was optimal for the translation bounds. Instead of iterated laminates or coated spheres, they used the Marino and Spagnolo type structures Marino and Spagnolo (1969) (reinvented by Sigmund in the paper Sigmund (2000)) that consist of rectangular domains with special conductivities that make separation of variables possible in the homogenization equations. Reinterpreting their results slightly, we divide the cell of periodicity into four rectangular subdomains. The opposite squares are occupied by K_2 and K_3, and the remaining rectangles are filled with laminates from K_1 and K_3 (see Figure 1d). The effective conductivity of the laminate depends on the volume fraction of materials in it. This conductivity (or, equivalently, the fraction of the materials in the laminate) is chosen in such a way that the conductivity equation (2) permits a separation of the variables if the external fields are homogeneous. Because of this feature, the solution is analytic and so are the effective properties. Using Maple, the authors then found that the structures are optimal for the translation bound (8). The authors also described more complicated structures that were optimal for larger values of m_1 and which coincided with the previously known structures at the point $K^* = K_2$.

4.2. **Optimal laminates.** Here we describe a new family of optimal microstructures for the case $N = 3$. They are the laminates with the special property that the fields inside the layers of pure material satisfy the sufficient conditions (P1)-(P3). We observe that they also necessarily satisfy the applicability condition (10). To find an optimal structure, we assign the fields in layers to be optimal and choose the volume fractions to allow compatibility for lamination. We begin with some degenerate cases and work toward the general structure.
A parameterization. The phase K_1 and its volume fraction m_1 play a special role in the bound (8) and in the associated optimal structures. For this reason, it is convenient to introduce the fraction p of K_2 relative to K_3,

$$p = \frac{m_2}{1 - m_1}, \quad 1 - p = \frac{m_3}{1 - m_1}.$$

Using p-notation, the translation bound (8) for three material mixtures is rewritten as

$$\frac{1}{2} \cdot \frac{\operatorname{tr} K^* - 2k_1}{\det K^* - k_1^2} \leq \frac{m_1}{2k_1} + (1 - m_1)\left(\frac{p}{k_2 + k_1} + \frac{1 - p}{k_3 + k_1}\right). \tag{11}$$

If we think of $p \in [0, 1]$ as a parameter of the problem, we can write the requirement that a structure attains bound (8) as

$$m_1 = \frac{\frac{1}{2} \cdot \frac{\operatorname{tr} K^* - 2k_1}{\det K^* - k_1^2} - \left(\frac{p}{k_2+k_1} + \frac{1-p}{k_3+k_1}\right)}{\frac{1}{2k_1} - \left(\frac{p}{k_2+k_1} + \frac{1-p}{k_3+k_1}\right)}. \tag{12}$$

Furthermore, the "coating principle" discussed in this section is an operation on structures which increases m_1, preserves p, and preserves the equality in (12). For this reason, it is convenient to fix p and plot the values of K^* where the bound (8) is sharp in the eigenvalue plane. From these values, we can recover via (12) the value of m_1 (and thus all other volume fractions) for each plotted point.

The lamination formula. The effective properties tensor K^* of a laminate from two anisotropic materials with conductivity tensors A and B, in volume fractions m and $1-m$ respectively, and with normal n to the layers, is given by the representation (see, for example Cherkaev (2000); Milton (2002))

$$K^* = L(K_A, K_B, n, m) = mK_A + (1-m)K_B - \mathcal{N} \tag{13}$$

where

$$\mathcal{N} = m(1-m)(K_B - K_A)n[n^T(mK_B + (1-m)K_A)n]^{-1}n^T(K_B - K_A).$$

Coating with $\mathbf{K_1}$ preserves optimality. In order to describe the variety of the optimal structures, we make the following observation.

Theorem 3 (The Coating Principle)**.** *If a structure K^* is optimal for the translation bound* (8)*, then all structures obtained by laminating it with material K_1 are also optimal for* (8)*. The laminating can be iterated so that the original structure is "coated" by K_1.*

This observation allows us to describe only the *extremal* structures that attain the bound (8) in the sense that they contain the minimal amount of K_1.

In particular, the coating principle immediately proves the optimality of all optimal two-phase structures – the laminates of second rank. These structures are the result of the coating of the pure phase K_2 (which is trivially optimal for (8)). The two-phase structures correspond to $p = 1$ (see (11) and (12)).

The coating principle also plays an important role in the analysis of multiphase mixtures. Notice that the coating changes the volume fractions, m_i, but it preserves the value of p. Since coating increases the value of m_1, the principle allows to look for the optimal structures with the lowest value of m_1. Every optimal structure generates a set of optimal coated structures.

T-structures. The simplest optimal structure is the T*-structure.* It is assembled as a sequence of laminates. First, K_1 and K_3 are laminated with normal in the x_1-direction. Then, the resulting composite is laminated with K_2 with the normal in the x_2-direction. Figure 2a illustrates the construction of the

T-structure. The effective properties of the T-structure are found by iterating the lamination formula for two materials K_A and K_B with normal n and in relative amounts m and $1-m$ respectively,

$$K_T = L\left(K_2,\ L\left(K_1, K_3, n_1, \frac{m_1}{m_1+m_3}\right),\ n_2, m_2\right)$$

where $n_1 = (1,0)^T$ and $n_2 = (0,1)^T$.

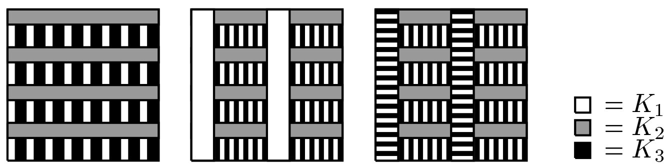

FIGURE 2. A selection structures optimal for (8). (a) a T-structure, (b) a T-structure with one layer of "coating", (c) a T^2-structure.

Theorem 4. *For all values of $p \in [0,1]$, there exists a T-structure with the given value of p that is optimal for the translation bound* (8).

It may seem surprising that we have found that there is always an optimal T-structure for any p. Keep in mind that we consider structures with fixed relative volume fractions of K_2 and K_3 but with arbitrary fraction of K_1.
Coated **T**-structures. From the optimal T-structure, we obtain a set of optimal structures by coating with K_1 according to Theorem 3. The obtained region is shaded in the eigenvalue plane in Figure 3. The calculation corresponds to the parameters

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 5, \quad p = \frac{1}{60}. \tag{14}$$

It is convenient to represent an anisotropic material by two symmetric points (λ_1, λ_2) and (λ_2, λ_1) in the plane of eigenvalues to avoid ordering the eigenvalues. Particularly, the optimal T-structure is represented by two points, both labeled K_T. The domain optimal structures given by coating the T-structure is the union of two lens-shaped regions in the plane. The boundaries of this set are the laminate curves. Recall that rather than fixing volume fractions, we fix the value p which in turn fixes the ratio of m_2 to m_3. The figure also includes some dotted curves of constant volume fraction. Those closer to K_1 indicate larger values of m_1 than those farther away. Any point where one of these curves intersects the region of optimal coated T-structures is an optimal point for the translation bound (8) with the volume fractions given through m_1 and p.

An extremely anisotropic T-structure with an additional layer of K_1 instead of coating in the x-direction is shown in Figure 2b. We notice that

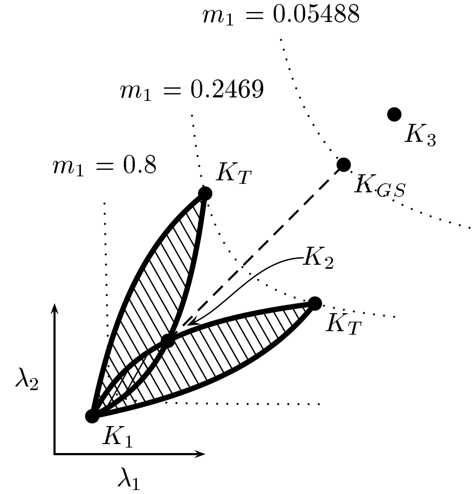

FIGURE 3. The set of optimal structures formed by coating the optimal T-structure.

these optimal structures that attain the translation bound are topologically equivalent to the extremal structures that attain the Wiener bound, Figure 1e (see Cherkaev and Gibiansky (1996)). The structures differ by a single parameter: the relative fraction of K_1 in the inner layer. Although these two structures are both optimal, they are optimal for different bounds. Therefore, it is not yet known if the structures with intermediate values of the parameter are optimal for some generalized bound. They are not optimal for any of the bounds (7)-(9). However, these structure would give a fair approximation to the boundary of all optimal structures.

$\mathbf{T^2}$-structures. Next, we enlarge the class of optimal structures by considering a generalization of the T-structure. We laminate the T-structure with a laminate of K_1 and K_3 in the orthogonal direction as seen in Figure 1c. The effective tensor of such T^2*-structures* is found from the iterative procedure

$$\begin{aligned} K_{T2} &= L(K_T, K'_{13}, n_1, \omega_2), \quad K_T = L(K_2, K_{13}, n_2, \omega_1), \\ K'_{13} &= L\left(K_1, K_3, n_2, \nu'\right), \quad K_{13} = L\left(K_1, K_3, n_1, \nu\right). \end{aligned}$$

The properties depend on four structural parameters: $\nu, \nu', \omega_1, \omega_2$ that all vary in $[0,1]$ and subject to the constraint that fixes p. The T^2-structures form a class of optimal anisotropic structures between the T-structures and the isotropic structures of Gibiansky and Sigmund, as is stated the following theorem.

Theorem 5. *For all values of $p \in [0,1]$ there exists a family of T^2-structures optimal for the bound* (8) *and with variable anisotropy. The optimal parameters of optimal structures satisfy the relations*

$$\nu = \Theta, \quad \nu' = \omega_1 \Theta$$

where

$$\Theta = \frac{k_1(k_3-k_2)}{(k_2+k_1)(k_3-k_1)} \le \frac{1}{2} \tag{15}$$

and

$$(1-2p\Theta)\omega_1\omega_2 + p\Theta(\omega_1+\omega_2) - p = 0, \quad \frac{p(1-\Theta)}{1-p\Theta} \le \omega_1, \omega_2 \le 1. \tag{16}$$

This construction generates a curve of eigenvalue pairs of effective tensors that passes through T-structure and the isotropic point of Gibiansky and Sigmund.

The volume fractions in the optimal laminates are

$$m_1 = \Theta(\omega_1(1-\omega_2)+\omega_2(1-\omega_1)), \quad m_2 = \omega_1\omega_2, \quad m_3 = 1-m_1-m_2. \tag{17}$$

The effective tensors have eigenvalues λ_1, λ_2 parameterized by

$$\lambda_1 = \frac{\omega_1 k_2(k_1+k_3)+(1-\omega_1)k_3(k_1+k_2)}{\omega_2(k_1+k_3)+(1-\omega_2)(k_1+k_2)} \tag{18}$$

$$\lambda_2 = \frac{\omega_2 k_2(k_1+k_3)+(1-\omega_2)k_3(k_1+k_2)}{\omega_1(k_1+k_3)+(1-\omega_1)(k_1+k_2)}. \tag{19}$$

The T^2 structures are extreme in the sense that (10) is satisfied as equality everywhere in Ω_1.

The extreme values of ω_1 or ω_2 correspond to the T-structure. On the other hand, when $\omega_1 = \omega_2$, we obtain an isotropic structure whose volume fractions satisfy the equality $m_1 = 2\Theta(\sqrt{m_2} - m_2)$ (compare to Gibiansky and Sigmund (2000) formula (49)!). The T^2-structures are a generalization of both the T-structures and the Gibiansky-Sigmund structure with minimal amount of m_1. The parameters ω_1, ω_2, ν and ν' are determined based on the requirements (P1)-(P3) on the fields inside the phases of an optimal structure.

Recall that we do not fix m_2. Instead we fix $p = \frac{m_2}{1-m_1}$ and allow the volume fractions to vary subject to this constraint. Using this, the natural limits $0 \le \omega_1, \omega_2 \le 1$ and the values of the volume fractions given in (17), we find (16).

The relation between the effective properties of optimal mixtures is symmetric to the interchanging $\omega_1 \leftrightarrow \omega_2$ in spite of the nonsymmetric iterative procedure.

The set of optimal structures. Combining the extremal T^2-structures with the coating principle, we obtain a variety of optimal structures because each T^2-structure can be coated, increasing the amount m_1 but keeping relative fraction p. The set of all coated T^2-structures with the given value of p forms a subset of structures optimal for the translation bound (8). This set is illustrated in the eigenvalue plane in Figure 4a for parameters as in (14). The set of optimal structures is bounded by the solid boundary, which is the union of coated T-structures (the curves between K_1 and K_T) and the T^2-structures (the curve passing through K_{GS}). The closed region bounded by the dotted lines represent the previously known optimal structures of Kohn and Milton, and Gibiansky and Sigmund.

5. Discussion

Curves of constant volume fraction are indicated in Figure 4a by the dotted lines. In particular the curve passing through K_T represents the case

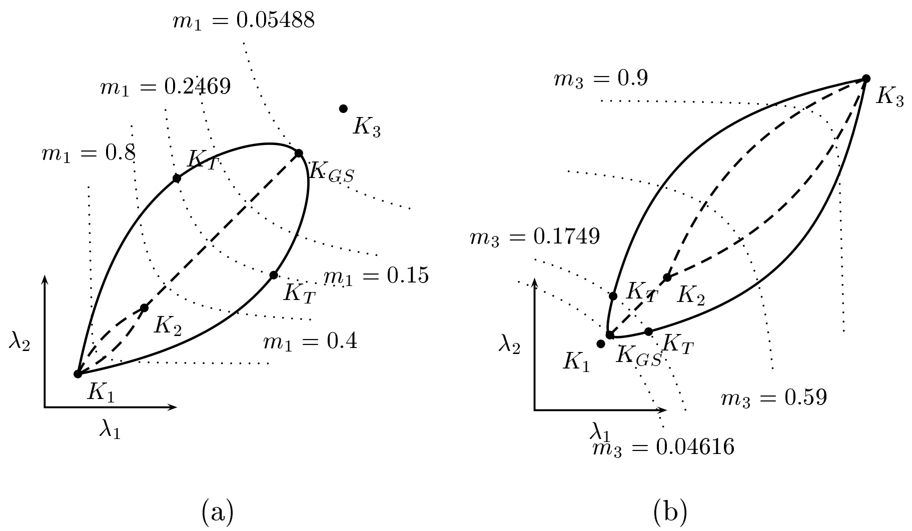

Figure 4. (a) Optimal points for the lower translation bound for $\frac{m_2}{1-m_1} = \frac{1}{60}$. (b) Optimal points for the upper translation bound for $\frac{m_2}{1-m_3} = \frac{1}{41}$.

$m_1 = \Theta(1 - m_2)$ while the curve passing through K_{GS} represents the case $m_1 = 2\Theta(\sqrt{m_2} - m_2)$.

Our results can be rewritten for the fixed volume fractions instead of fixed parameter $p = m_2/(1 - m_1)$. For example, consider the question of the G-closure when

$$m_1 = 0.4, \quad m_2 = 0.01, \quad m_3 = 0.59.$$

We need only examine Figure 4a and the corresponding figure for the upper bound, Figure 4b. The optimal points of the lower bound marked by the thick dashed curve in Figure 5 are the intersection of the curve of constant $m_1 = 0.4$ with the optimal region shown in Figure 4a. The dot in Figure 5 marks the point where this curve intersects the dashed Gibiansky-Sigmund line. Similarly, the optimal points marked by the thick curve on the upper bound in Figure 5 are points where the line of constant volume fraction $m_3 = 0.59$ intersects the set of optimal points in Figure 4b. The solid portion of the curve marks the intersection with the Kohn-Milton region.

Look again at Figure 4a. As long as $m_1 \geq 2\Theta(\sqrt{m_2} - m_2)$, the intersection of the constant m_1 curve and the region of optimal points is a connected

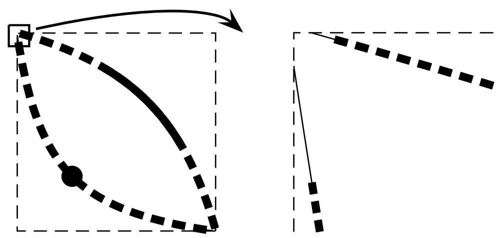

FIGURE 5. (left) The bounds (7)-(9) with known optimal points indicated by thick lines. (right) a magnification of the upper-left corner.

section of the curve which includes the isotropic point. Thus, the intersection can be described by the most anisotropic point only. For $m_1 > \Theta(1-m_2)$ this most anisotropic point is a coated T-structure. For $m_1 \le \Theta(1-m_2)$, it is a T^2-structure. We summarize this in the following theorem.

Theorem 6. *Let the volume fractions m_1, m_2 and m_3 be given.*

(i) If $m_1 > \Theta(1-m_2)$, then (8) is sharp. Moreover, there exists a set of optimal tensors whose most anisotropic member is that given by the optimal coated T-structure with the given volume fractions.

(ii) If $2\Theta(\sqrt{m_2}-m_2) \le m_1 \le \Theta(1-m_2)$, then (8) is sharp. Moreover, there exists a set of optimal tensors whose most anisotropic member is that given by the optimal T^2-structure satisfying the volume fraction constraints. In particular, the parameters ω_1 and ω_2 for this most anisotropic structure can be found by solving simultaneously the equations

$$m_1 = \Theta\omega_1(1-\omega_2) + \Theta\omega_2(1-\omega_1), \quad m_2 = \omega_1\omega_2.$$

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