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Minimax optimization problem of structural design

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Abstract

The paper discusses a problem of robust optimal design of elastic structures when the loading is unknown. It is assumed that only an integral constraint for the loading is given. We suggest to minimize the principal compliance of the domain equal to the maximum of the stored energy over all admissible loadings. The principal compliance is the maximal compliance under the extreme, worst possible loading. Hence the robust design should optimize the behavior of the structure in the worst possible scenario, which itself depends on the structure and is subject of optimization. We formulate the problem of robust optimal design as a min–max problem for the energy stored in the structure. The maximum of the energy is chosen over the constrained class of loadings, while the minimum is taken over the set of design parameters. We show that the problem for the extreme loading can be reduced to an elasticity problem with mixed nonlinear boundary condition; this problem may have multiple solutions. The optimization with respect to the designed structure takes into account the possible multiplicity of extreme loadings so that in the optimal design the strong material is distributed to equally resist to all extreme loadings. Continuous change of the loading constraint causes bifurcation of the solution of the optimization problem. We show that an invariance of the constraints under a symmetry transformation leads to a symmetry of the optimal design. Examples of robust optimal design are demonstrated.

Keywords: Structural design; Bifurcation; Steklov eigenvalues; Minimax; Robust design; Constrained optimization

1. Introduction

Structural optimization is a problem of distributing given materials in the structure to create a stiffest design. If the applied external force is given, the optimally designed structure minimizes the elastic energy of a domain. However, the optimal designs are usually unstable to variations of the forces. This instability is a direct result of optimization: To best resist the given loading, the structure concentrates its ability to resist the loading in a certain direction thus decreasing its ability to sustain loadings in other directions [8,9,19]. For example, consider a problem of optimal design of a structure of a cube of maximal stiffness made from an elastic material and void; assume that the cube is supported on its lower side and loaded by a homogeneous vertical force on its upper side. It is easy to demonstrate, that the optimal structure is a periodic array of unconnected infinitely thin cylindrical rods. Obviously, this design does not resist any other but the vertical loading.

To avoid this vulnerability of the optimally designed structures to variations of loading, we suggest to minimize *the principal compliance* of the domain equal to the maximum of the stored energy over all admissible loadings. The principal compliance is the maximal compliance under the extreme, worst possible loading. We formulate the robust optimal design problem as a min–max problem for the energy stored in the domain, where the inner maximum is taken over the set of admissible loadings and the minimum is chosen over the design parameters characterizing the structure. This formulation corresponds to physical situations when the loadings are not known in advance,

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such as in construction of engineering structures or biological materials.

This approach to the structural optimization was discussed in our papers [13,11,12] and (for the finite-dimensional model) in the papers [21,22]. Various aspects of the optimal design against partly unknown loadings were studied in [33,23,32,28,27,38,5,8,26,1,7], see also references therein. In some cases, the minimax design problem, where the designed structure is chosen to minimize maximal compliance of the domain, can be formulated as minimization of the largest eigenvalue of an operator. The minimization of dominant eigenvalues was considered in a setting of inverse conductivity problem in [14,15]. The multiplicity of optimal design that we find in the minimax loading-versus-design problem is similar to multiplicity of stationary solutions investigated in the engineering problems of the optimal design against vibration [31,34,24,29] and buckling [35,16].

The introduced *principal compliance* [12], is an integral characteristic of an elastic domain, equal to the response of the domain to the worst (extremal) boundary force from the given class of loadings; this quantity is a basic characteristic of the domain similar to the capacity, principal eigenfrequency, or volume. The principal compliance is a solution of a variational problem, which can be reduced to an eigenvalue problem, or to a bifurcation problem. We discuss this in Section 2.

Examples of constraints for admissible loadings and corresponding variational problems are considered in Section 3. Particularly, the variational problem for the principal compliance with a quadratically constrained class of loadings is reduced to the Steklov eigenvalue problem. The principal compliance of the domain in this case is the reciprocal of the first Steklov eigenvalue. The optimal loading in the class of forces with the constrained L_1 norm, is a concentrated loading (if such a loading does not lead to infinite energy). Other constraints such as for the L_p norm, $p > 1$, of the loading and inhomogeneous constraints are considered in [12], it is shown that the L_p norm constraints result in a nonlinear boundary value problem.

Section 4 considers robust structural optimization which is formulated as a problem of minimization of the principal compliance. The optimal design takes into account the multiplicity of stationary solutions for extreme (most dangerous) loadings; typically, the optimal structure equally resists several extreme loadings. The set of the extreme loadings depends on the constraints of the problem. Continuous change of the constraints leads to modification of the set of extreme loadings; the optimal structure is changing in response. This corresponds to bifurcation of the solution of the optimization problem. Another characteristic feature of the discussed optimization problem is symmetry of its solution. The invariance of the set of the constraints for the admissible loadings together with the corresponding symmetry of the domain, leads to the symmetry of the optimally designed structure [12].

Sections 5 and 6 contain two examples of problems of structural design for uncertain loadings. One example is design of the optimally supported beam loaded by an unknown force with fixed mean value. The second example is a problem of determining the optimal structure of a composite strip loaded by a force deviated from the normal in an unknown direction. The force is assumed to have a prescribed normal component and an additional component which is arbitrarily directed and is unknown.

2. The principal compliance of a domain

Consider a domain Ω with the boundary $\partial\Omega = \partial_0 \cup \partial$ filled with a linear anisotropic elastic material, loaded from its boundary component ∂ by a force f, and fixed on the boundary component ∂_0. The elastic equilibrium of such a body is described by a system (see for instance, [36]):

$$\nabla \cdot \sigma = 0 \quad \text{in } \Omega, \qquad \sigma = C : \epsilon, \sigma = \sigma^{\mathrm{T}}, \\ \epsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^{\mathrm{T}}). \tag{1}$$

Here $C = C(x)$ is the fourth-order stiffness tensor of an anisotropic inhomogeneous material, $w = w(x)$ is the displacement vector, ϵ is the strain tensor, σ is the stress tensor, and (:) is convolution of two indices. The above convolutions read:

$$\epsilon : \sigma = \sum_{i,j} \epsilon_{ij}\sigma_{ji}, \qquad (C : \epsilon)_{ij} = \sum_{k,l} C_{ijkl}\epsilon_{lk}.$$

Eq. (1) is supplemented with the boundary conditions

$$\sigma \cdot n = f \quad \text{on } \partial, \qquad w = 0 \quad \text{on } \partial_0, \tag{2}$$

where n is the normal to the boundary $\partial\Omega$. These equations are the stationary solution of a variational problem,

$$\begin{aligned} \mathscr{J}(C,f) &= -\min_{w: w|_{\partial_0}=0} \left(\int_\Omega \Pi(C,\epsilon(w))\,\mathrm{d}x - \int_\partial w \cdot f\,\mathrm{d}s \right) \\ &= \max_{w: w|_{\partial_0}=0} \left(\int_\partial w \cdot f\,\mathrm{d}s - \int_\Omega \Pi(C,\epsilon(w))\,\mathrm{d}x \right), \end{aligned} \tag{3}$$

where Π is the density of the elastic energy:

$$\Pi(C,\epsilon(w)) = \frac{1}{2}\epsilon : \sigma = \frac{1}{2}\epsilon : C : \epsilon. \tag{4}$$

The nonnegative functional $\mathscr{J}$ is called the *compliance* of the domain; (3) states that it is maximal at the elastic equilibrium. At the equilibrium, the energy stored in the body equals the work of the applied external forces f,

$$\mathscr{J}_0(C,f) = \frac{1}{2}\int_\partial w \cdot f\,\mathrm{d}s = \int_\Omega \Pi(C,\epsilon(w))\,\mathrm{d}x. \tag{5}$$

Simultaneously with the elasticity problem, we consider also a close problem of bending of a Kirchhoff plate (see for example, [36]). The equilibrium of the plate is described by the fourth-order equation

$$\nabla\nabla : C_{pl} : \nabla\nabla w = f \quad \text{in } \Omega \tag{6}$$

with homogeneous boundary conditions

$$w = 0 \quad \text{on } \partial\Omega, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

corresponding to a clamped plate, or

$$w = 0 \quad \text{on } \partial\Omega, \qquad n^{\mathrm{T}}(C_{pl} : \nabla\nabla w)n = 0 \quad \text{on } \partial\Omega, \tag{8}$$

for simply supported plate. Here, w is the deflection orthogonal to the plane of the plate, C_{pl} is the fourth-order tensor of bending stiffness of the elastic material, $\nabla\nabla w$ is the Hessian of w, and f is the external loading. Notice that the force f enters the equation as a right-hand-side term. The equation of the plate corresponds to maximization of the functional:

$$\mathscr{J}_{pl}(C,f) = -\int_{\Omega}\left(\frac{1}{2}\nabla\nabla w : C_{pl} : \nabla\nabla w - wf\right)\mathrm{d}x. \tag{9}$$

The results that we develop further in the paper apply to both the elasticity (1) and the bending problem (6); therefore, we will drop the subscript in $\mathscr{J}_{pl}(C,f)$, and keep notation $\mathscr{J}(C,f)$ for both compliance functionals. If this does not cause confusion, we use the same notation w to denote both the displacement in the elasticity problem (1) and the deflection in the bending problem (6), in spite of the first one being a vector function, whereas the second one is a scalar function.

2.1. Admissible loadings

Let $\mathscr{F}$ be a set of admissible loadings f. The elastic energy over a finite domain is assumed to be finite. We consider integral constraints to describe the set of loadings $\mathscr{F}$:

$$\mathscr{F} = \left\{f : \int_{D_f}\phi(f)\,\mathrm{d}s = 1\right\},$$

$$D_f = \begin{cases}\partial, & \text{for problem (1)},\\ \Omega, & \text{for problem (6)},\end{cases} \tag{10}$$

Here D_f is a domain of application of the forces: In the elasticity problem (1), D_f coincides with the part of the boundary ∂, whereas for the bending plate problem (6), D_f is the domain Ω or a part of it. We assume that ϕ is a convex function of f, with the derivative $\psi : R^3 \to R^3$:

$$\psi(f) = \frac{\partial\phi}{\partial f} = \left(\frac{\partial\phi}{\partial f_1}, \frac{\partial\phi}{\partial f_2}, \frac{\partial\phi}{\partial f_3}\right),$$

which has an inverse $\rho = \psi^{-1}$.

2.2. Principal compliance

We define the *principal compliance* of an elastic domain in a class of loadings as a compliance in the worst possible loading scenario.

Definition. The principal compliance Λ of the domain is

$$\Lambda = \max_{f\in\mathscr{F}}\mathscr{J}(C,f). \tag{11}$$

The forces that correspond to the principal compliance Λ are the *extreme* or *the most dangerous loadings*; we denote them as f_D.

$$\Lambda(C) = \mathscr{J}(C,f_D) \geqslant \mathscr{J}(C,f) \quad \forall f \in \mathscr{F}. \tag{12}$$

Consider problem (11) and assume that the loadings are constrained as in (10). The augmented functional J for the problem is

$$J = \mathscr{J}(C,f) - \mu\left(\int_{D_f}\phi(f)\,\mathrm{d}s - 1\right),$$

where μ is the Lagrange multiplier. Clearly, $\max_{f\in\mathscr{F}}\mathscr{J} = \max_f J$. Variation of the augmented functional with respect to f gives the optimality condition for the extreme loading(s):

$$\delta_f J = \int_{D_f}\frac{\partial}{\partial f}(-f\cdot w + \mu\phi(f))\delta f = 0,$$

or, since δf is arbitrary,

$$w - \mu\frac{\partial\phi}{\partial f} = 0 \quad \text{on } D_f.$$

Solving for the extreme loading(s) $f_D = f$ we arrive at the condition

$$f_D = \rho\left(\frac{w}{\mu}\right), \tag{13}$$

which links the loading f_D to the displacement w at the same boundary point for the elasticity problem (1) or at the same point in the domain for the bending problem. Condition (13) together with the first boundary condition in (2) allows us to exclude f from the boundary conditions, leading to the boundary value problem for the displacement w. This results in the following problem for the principal compliance. The principal compliance Λ of the elasticity problems (1 and 2) with the constraint for the class of loadings (10) equals

$$\Lambda = \frac{1}{2}\int_{\partial} w\rho\left(\frac{w}{\mu}\right)\mathrm{d}s, \tag{14}$$

where w satisfies the elasticity equation (1) in Ω with the boundary conditions

$$\sigma\cdot n = \rho\left(\frac{1}{\mu}w\right) \quad \text{on } \partial, \quad w = 0 \quad \text{on } \partial_0, \tag{15}$$

The Lagrange multiplier μ is determined from the integral condition

$$\int_{\partial}\phi\left(\rho\left(\frac{w}{\mu}\right)\right)\mathrm{d}s = 1, \tag{16}$$

and the function $\rho(\cdot)$ is an inverse of $\psi = \frac{\partial\phi}{\partial f}$.

For the bending problem (6), the derivation is similar. The principal compliance is the maximum of the functional (9) upon all loadings bounded by the constraint (10), its value is the following. The principal compliance Λ for the bending problems (6)–(8) with the constraint for the class of loadings (10) equals

$$\Lambda = \frac{1}{2}\int_\Omega w\rho\left(\frac{w}{\mu}\right)\mathrm{d}x, \tag{17}$$

where w satisfies the equation

$$\nabla\nabla : C_{pl} : \nabla\nabla w = \rho\left(\frac{w}{\mu}\right) \tag{18}$$

with the corresponding homogeneous boundary conditions (7) or (8), and the function $\rho(\cdot)$ being an inverse of $\psi = \frac{\partial\phi}{\partial f}$. The Lagrange multiplier μ is determined from

$$\int_\Omega \phi\left(\rho\left(\frac{w}{\mu}\right)\right)\mathrm{d}s = 1. \tag{19}$$

Indeed, the extreme loading f is related to the displacement w by a scalar relation $w = \mu\phi'(f)$ or $f = \rho(w/\mu)$, and the plate equilibrium is described by Eq. (18).

3. Examples of constraints

3.1. Homogeneous quadratic constraint

Assume that the constraint (10) restricts a weighted L_2 norm of f:

$$\frac{1}{2}\int_\partial f^{\mathrm{T}}\Psi f\,\mathrm{d}s = 1 \quad \text{or} \quad \phi(f) = \frac{1}{2}f^{\mathrm{T}}\Psi f, \tag{20}$$

where $\Psi(s)$ is a symmetric positive matrix. In this case, ρ is a linear mapping: $\rho(f) = \Psi^{-1}f$, and the first of the boundary conditions (15) for the extremal loading becomes linear:

$$\frac{1}{\mu}\Psi^{-1}w - \sigma\cdot n = 0 \quad \text{on } \partial. \tag{21}$$

The optimality condition states that w and $\sigma\cdot n$ are proportional to each other everywhere on the boundary ∂ with the same tensor of proportionality $\mu\Psi$.

The elasticity equation (1) with boundary conditions (21) form a linear *eigenvalue problem* that has a nonzero solution w only if $\frac{1}{\mu}$ is one of its discrete eigenvalues [37]. Eigenvalue $\frac{1}{\mu}$ relates the displacement on the boundary and the normal stress. As an eigenvalue problem, the problem (1) with the boundary conditions (21) is an Euler–Lagrange equation of a variational problem:

$$\frac{1}{\mu} = \min_{w:w|_\partial = 0}\frac{\int_\Omega \epsilon(w) : C : \epsilon(w)\mathrm{d}s}{\int_\partial w\cdot\Psi^{-1}w\,\mathrm{d}s}$$

or

$$\left(\int_\Omega \epsilon(w) : C : \epsilon(w)\mathrm{d}x - \frac{1}{\mu}\int_\partial w\cdot\Psi^{-1}w\,\mathrm{d}s\right) \to \min_{w:w|_\partial = 0}. \tag{22}$$

The eigenvalue problem that contains the eigenvalue in the boundary condition is *Steklov eigenvalue problem*, and μ is a reciprocal to the Steklov eigenvalue, see [4]. The eigenfunctions are normalized by the condition (20).

Using (20) and (21) in the form $w = \mu\Psi f$, we observe that the second term in (22) is equal to μ, thereafter $\mu = \Lambda$. Steklov problem has infinitely many real positive eigenvalues (see [4,25]), but the principal compliance of the domain corresponds to the dominant eigenvalue, $\Lambda = \mu_{\max}$. The dominant eigenfunction is not necessarily unique; we will demonstrate below that the existence of many stationary solutions is typical for the problems of minimization of the principal compliance with respect to the structure. The dominant eigenfunctions are the extreme loadings. If the L_2-norm of admissible loadings is bounded, the principal compliance Λ is a solution of the eigenvalue problem:

$$\nabla\cdot\sigma = 0 \quad \text{in } \Omega, \qquad w = \Lambda\Psi\sigma\cdot n \quad \text{on } \partial. \tag{23}$$

Λ is a reciprocal to the principal eigenvalue $\frac{1}{\mu}$ of the problem (1), (21).

The problem becomes isomorphic to the problem of the principal eigenfrequency of the domain, if the kinetic energy (and the inertia) are concentrated on the boundary: $T = \delta(x - x_b)w\Psi w$, where $x_b \in \partial$. In the bending problem (6), the analogy between the principal compliance and the principal eigenfrequency of vibrations is complete. The equilibrium (18) of the optimally loaded plate coincides with the equation for the magnitude of the deflection of the oscillating plate,

$$\nabla\nabla : C_{pl} : \nabla\nabla w = \frac{1}{\Lambda}w.$$

3.2. L_1-norm constraint

Consider L_1-norm constraint for the class of admissible loadings assuming that the mean value of loading's magnitude is fixed:

$$\int_\partial |f|\,\mathrm{d}s = \int_\partial \sqrt{f\cdot f}\,\mathrm{d}s = 1. \tag{24}$$

From engineering viewpoint, this case is probably the most interesting one: it models the situation when the total weight applied to the structure is known but the distribution of the loading over the boundary is uncertain. To avoid singularity using L_1-constraint, we may constrain the $L_{1+\epsilon}$-norm of the loading,

$$\int_\partial |f|^{1+\epsilon}\,\mathrm{d}s = 1, \tag{25}$$

where $\epsilon > 0$ is a fixed parameter. This loading can be supported by a linear elastic material, although the displacement w can indefinitely grow when $\epsilon \to 0$. The analysis of this case gives the optimality condition

$$f = \left|\frac{w}{\mu}\right|^{1/\epsilon}\frac{w}{|w|}$$

that shows that magnitude of an optimal force either stays arbitrary close to zero or is very large (of the order of $1/\epsilon$).

The integral constraint (25) guarantees that the measure of the set of large values of $f(s)$ goes to zero when $\epsilon \to 0$.

The extremal loading is concentrated in several points,

$$f = \sum_i c_i \xi_i \delta(x - x_i),$$

where $\{x_i\}$ is the set of points where the force is applied, $x_i \in \partial$, $\xi_i : \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}), |\xi_i| = 1$, are directional vectors of the concentrated loadings, and c_i are their intensities; due to (24), c_i belong to the simplex

$$c_i : \sum_i c_i = 1, \quad c_i \geqslant 0. \tag{26}$$

Further, we show in [12] that the extreme loading is always applied to a single point. The L_1-principal compliance is

$$\Lambda = \max_{x \in \partial} \{\lambda^g_{\max}(x)\}.$$

where $\lambda^g_{\max}(x)$ is the maximal eigenvalue of the 3×3 tensor Green's function $g(x, x)$ of the problem (1) in the point $x \in \partial$.

4. Robust structural optimization

Robust structural optimization is formulated as a problem of minimization of the principal compliance. The optimal design takes into account the multiplicity of stationary solutions for extreme (most dangerous) loadings resulting in the optimal structure which equally resists several extreme loadings. Consider an optimal design problem: Find a layout of elastic materials over the domain that minimizes the principal compliance Λ. Such a structure (stiffness $C(x)$) corresponds to a solution of the extremal problem

$$P_{\min\max} = \min_{C \in \mathscr{C}} \Lambda(C), \tag{27}$$

where $\mathscr{C}$ is a class of admissible layouts.

We can show that the optimization problem is reduced to minimization of a weighted sum of principal compliances. The optimal principal compliance $P_{\min\max}$ equals

$$P_{\min\max} = \min_{C \in \mathscr{C}} \max_{\{v_i\}: v_i > 0} \sum_{i=1}^{q} v_i \mathscr{J}(C, \hat{f}_i), \quad \sum_i v_i = 1, \tag{28}$$

where q is the number of active extreme loadings. This problem admits a probabilistic interpretation. Assume that the optimal loading is a random variable which takes q stationary values with some probability $v_1, \ldots, v_q$. Then the sum $\sum v_i \mathscr{J}(C, f_i)$ in (28) is the expectation of the energy. The optimal design minimizes the expectation of the energy, meanwhile the loading chooses probabilities $v_1, \ldots, v_q$ to maximize it.

Symmetry of the design that minimizes the principal compliance is a characteristic feature of the optimal design, which follows from multiplicity of optimal (extreme) loadings. If the domain and the class of loadings are invariant under a symmetry transformation (translation, reflection, or rotation), then the set of extreme loadings Φ and the optimal design are invariant under this transformation as well. We state the following theorem in [12]. If the domain Ω, the boundary component ∂, and the set $\mathscr{F}$ of admissible loadings are invariant under a symmetry transformation $\mathscr{R}$: $\Omega = \mathscr{R}\Omega$, $\partial = \mathscr{R}\partial$, and $\mathscr{F} = \mathscr{R}\mathscr{F}$, then the set of extreme loadings Φ and the optimal materials' layout C are invariant under this transformation: $\Phi = \mathscr{R}\Phi, C = \mathscr{R}C$.

The characteristic feature of the optimization problem is multiplicity of equally dangerous loadings. This closely reminds the multiplicity of optimal solutions in a problem of maximization of the minimal eigenfrequency [18]. First, the multiplicity of optimal eigenvalues in that problem was observed in a pioneering paper of Olhoff and Rasmussen [31], then it was investigated in [16,34,35]. Below we show two examples which highlight and illustrate multiplicity of extreme loadings and bifurcation of the optimal solution.

5. Optimal design of a supported beam

Consider a homogeneous elastic beam of the unit length simply supported at both ends (see Fig. 1), elastically supported from below by a distributed system of elastic vertical springs with the specific stiffness $q(x) \geqslant 0$, and loaded by a distributed nonnegative force $f(x) \geqslant 0$. The elastic equilibrium of the displacement w is described by a one-dimensional version of (6):

$$\begin{aligned} &(Ew'')'' + qw = f, \quad w(0) = w(1) = 0, \\ &w''(0) = w''(1) = 0, \end{aligned} \tag{29}$$

where E is the Young's modulus. The compliance is equal to

$$\mathscr{J} = \int_0^1 \left(fw - \frac{E}{2}(w'')^2 - \frac{q}{2}w^2 \right) \mathrm{d}x, \tag{30}$$

where w is a solution to (29). Assume that the mean value of the magnitude of the loading (L_1-norm constraint) is equal to one, and the integral stiffness of the supporting springs is constrained by a constant κ.

$$\mathscr{F} = \left\{ f \in H^{-1}(0,1) : \int_0^1 f \, \mathrm{d}x = 1 \right\},$$
$$\mathscr{Q} = \left\{ q \in H^{-1}(0,1) : \int_0^1 q \, \mathrm{d}x = \kappa \right\}.$$

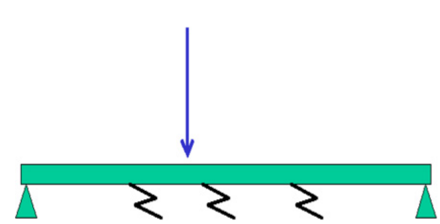

Fig. 1. The force could be applied in arbitrary points along the elastically supported beam. The mean value of the magnitude of the force is constrained.

The optimal design problem of minimization of the principal compliance by distributing the springs stiffness becomes:

$$P_{\min\max} = \min_{q\in\mathcal{Q}}\left(\max_{f:\in\mathcal{F}}\mathcal{J}\right).$$

Analysing the problem we conclude that the domain, class of loadings and the boundary conditions are invariant to the translation $x \to 1-x$, therefore, the design (the springs stiffness) is symmetric with respect to the center of the beam,

$$q(x) = q(1-x).$$

Necessary conditions of optimality show, that the extreme loading is a delta-function $f(x) = \delta(x - x_j)$ applied at one of the points $\{x_1, x_2, \ldots, x_p\}$ where the value of w is maximal. The extreme loading may be applied to different points symmetric with respect to the center of the beam; the resulting stiffness must be equal. The stiffness of an optimal spring is a distribution

$$q(x) = \sum_i \alpha_i\delta(x - y_i), \quad \sum_i \alpha_i = \kappa, \quad \alpha_i \geqslant 0.$$

Particularly, the optimal positions of the springs satisfy the necessary conditions of optimality, and therefore the set of the reinforcement points coincides with the set $\{x_1, x_2, \ldots, x_p\}$. The number p of the critical points depends on the relative stiffness of the springs κ/E.

Accounting for the loading and springs being concentrated, we reformulate the problem (30) for the optimal principal compliance:

$$P_{\min\max} = \min_{\alpha_1,\ldots,\alpha_p}\max_{x_k}\left\{\sum_{i=1}^{p}\left(\delta_{ik}w_k - \frac{\alpha_i}{2}w_i^2\right) - \int_0^1 \frac{E}{2}(w'')^2\,\mathrm{d}x\right\}, \tag{31}$$

where δ_{ik} is Dirac function.

The response of the beam due to a force moving along the beam, can be characterized by the curve, $v(x)$, showing the maximal displacement due to the force applied at the point x. Generally, the point of application of the concentrated force is different from the point of maximum of the displacement curve. However, for the optimal forces these points coincide.

The numerical results demonstrate the following: If the springs are weak, $\kappa/E \leqslant \kappa_1$, they are concentrated in the center of the beam. We are dealing with the saddle-point case: The most dangerous loading is a concentrated force applied also in the center (see top Fig. 3). The maximal displacement is a unimodal function of the position of the loading, with the maximum in the center. There is only one solution for the optimal applied force and the optimal position of the spring:

$$f(x) = \delta(x - 1/2), \quad q(x) = \kappa\delta(x - 1/2).$$

The top plot in Fig. 2 shows $v(x)$ for the beam supported by a weak spring in the center of the beam. One can see that

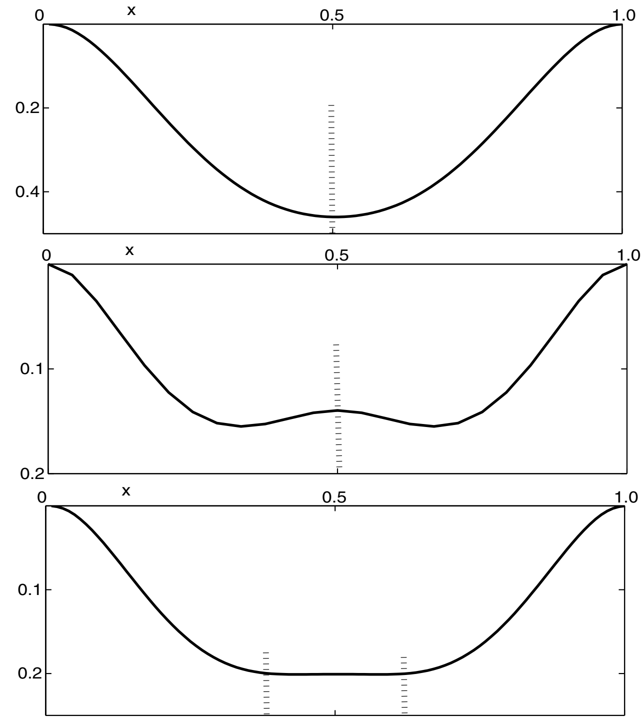

Fig. 2. Maximal displacement $v(x)$ as a function of the position of the applied loading. Top figure corresponds to a saddle-point case, $\kappa/E < \kappa_1$: The function $v(x)$ is unimodal, the optimal spring and the extreme loading are both located in the middle of the beam. Center figure shows $v(x)$ corresponding to κ/E in the interval $\kappa_1 < \kappa/E < \kappa_2$ when the strong spring is located in the center of the beam. Maximal displacement $v(x)$ is not unimodal; design is not optimal. Bottom figure corresponds to κ/E in the same interval $\kappa_1 < \kappa/E < \kappa_2$, the maximal displacement $v(x)$ is shown for optimally designed beam which is supported by two symmetric springs.

$v(x)$ is unimodal. The corresponding beam is shown in Fig. 3.

If the spring becomes stronger, $\kappa_1 < \kappa/E \leqslant \kappa_2$, but is still located in the center, the maximum of $v(x)$ corresponds to a noncentral applied force. The equally dangerous loadings could be applied in two symmetric eccentric points. The maximum displacement $v(x)$, shown in Fig. 2 (center), is not a unimodal function of the position of the moving applied force; the design is not optimal. The optimal design for this case (Fig. 2, bottom) corresponds to two equally

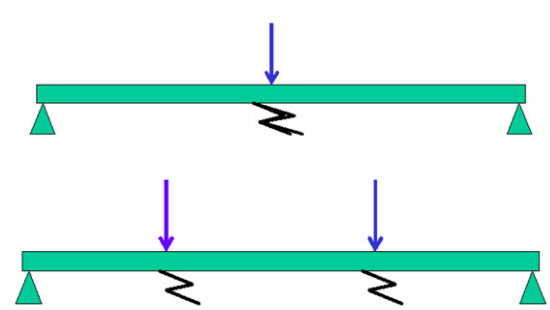

Fig. 3. Optimally supported beam for weak spring (top) and for strong spring (bottom): in this case two strong springs are located symmetrically with respect to the center of the beam.

stiff springs located symmetrically with respect to the center; the design experiences a bifurcation at the critical value of $\kappa/E = \kappa_1$.

Optimally supported beam is shown in Fig. 3 (bottom): two strong springs are located symmetrically to the center of the beam. The corresponding maximal displacement function $v(x)$ is shown in Fig. 2 (bottom). The maximal displacement curve becomes unimodal again, with a large interval of almost constant values in the middle.

Next bifurcation occurs when κ further increases, at the point $\kappa/E = \kappa_2$. Three springs appear after the next bifurcation. The number of optimal supporting points increases and tends to infinity when the springs are much stronger than the beam, $\kappa/E \gg 1$. The optimality conditions

$$w'(x_i) = 0, \quad w(x_i)|_{f=f_i} = \text{constant}(i)$$

give the optimal position of the supporting springs x_i and requirement to their stiffnesses α_i.

6. Design of composite strip for loading of uncertain deviation from the normal

Consider an infinite strip $\Omega = \{-\infty < x < \infty, -1 \leqslant y \leqslant 1\}$, made from a two-component elastic composite with arbitrary structure but with fixed fractions m_A and $m_B = 1 - m_A$ of the isotropic components. The stiffness of the composite $C(x, y)$ is an anisotropic elasticity tensor; it is assumed that the stiffness can vary only along the strip, $C = \text{constant}(y)$.

Assume that the upper boundary is loaded by some unknown but uniform loading f,

$$\sigma(x, 1) \cdot N = f \quad \forall x,$$

where $N = (0, 1)$ is the normal vector. Loading f consists of the fixed component $f_0 = (0, 1)$ directed along the normal and a variable component (deviation) (f_N, f_T), the magnitude of the deviation is constrained:

$$f = (f_0 + f_N)N + f_T T, \quad f_N^2 + f_T^2 = \gamma^2. \tag{32}$$

Here $T = (1, 0)$ is the tangent vector and γ is the intensity of the deviation. The constraint (32) can be rewritten as

$$f = (f_0 + \gamma\cos\theta)N + (\gamma\sin\theta)T \quad \text{for } y = 1,$$

where θ is the angle of inclination of the deviation of the loading, see Fig. 4. The lower boundary of the strip is assumed to be loaded by a symmetrically deviated force

$$f_- = -f = -(f_0 + \gamma\cos\theta)N + (\gamma\sin(-\theta))T \quad \text{for } y = -1.$$

The symmetry of the loadings results in the horizontal strain being zero,

$$\epsilon_{xx}(x, y) = 0, \quad -1 \leqslant y \leqslant 1, \tag{33}$$

so that the strain tensor has only two, vertical and shear, nonzero components. The stiffness of the composite $C(x)$ is an anisotropic tensor that is assumed to vary only along x coordinate. We consider the problem of optimization of the principal compliance of the described domain.

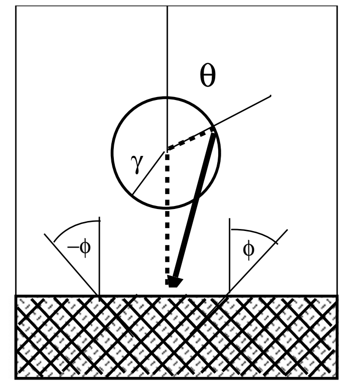

Fig. 4. An infinite composite strip loaded by a force f that could deviate from the normal direction. If the norm γ of the deviation is smaller than a critical value γ_1, the optimal composite is a laminate with layers directed across the strip. If γ is greater than γ_1, the optimal composite is second-rank laminate with layers oriented along directions ϕ and $-\phi$.

Applying the symmetry theorem, we conclude that the elastic properties of the optimally designed structure do not vary along the strip, since the design is invariant to translation $x \to x + \chi$. Together with the assumption that the material properties do not vary with the thickness, this leads to the conclusion that the elastic properties are uniform: the tensor C is constant of x and y. This implies that the stress field σ is constant inside an optimal strip and

$$\sigma_{yy} = 1 + \gamma\cos\theta, \quad \sigma_{xy} = \gamma\sin\theta. \tag{34}$$

The material in the optimal strip is orthotropic with main axes directed along x and y axes since the design is invariant to the reflection $x \to -x$. This implies orthotropy with the main axes codirected along x, y axes.

6.1. The optimization problem

The energy Π of an orthotropic material is computed either as a function of stresses and compliance tensor $S = \{S_{ij}\}$, (stress energy):

$$\Pi_\sigma(S, \sigma) = \frac{1}{2}(S_{11}\sigma_1^2 + S_{22}\sigma_2^2 + 2S_{12}\sigma_1\sigma_2 + 2S_{33}\sigma_3^2), \tag{35}$$

or as a function of strain ϵ and stiffness tensor $C = \{C_{ij}\}$,

$$\Pi_\epsilon(C, \epsilon) = \frac{1}{2}(C_{11}\epsilon_1^2 + C_{22}\epsilon_2^2 + 2C_{12}\epsilon_1\epsilon_2 + 2C_{33}\epsilon_3^2). \tag{36}$$

Two components $\sigma_1 = \sigma_{yy}$ and $\sigma_3 = \sigma_{xy}$ of the stress field σ are known (34), and the strain in the xx direction is zero, (33):

$$\epsilon_2 = S_{12}\sigma_1 + S_{22}\sigma_2 = 0;$$

therefore, σ_2 can be excluded. The elastic energy (36) becomes

$$\Pi_\epsilon(C, \epsilon) = \frac{1}{2}(C_{11}\epsilon_1^2 + 2C_{33}\epsilon_3^2)$$

or, in terms of stress (see (35)),

$$\Pi_\sigma(S,\sigma) = \frac{1}{2}\left(\left(S_{11} - \frac{S_{12}^2}{S_{22}}\right)\sigma_1^2 + 2S_{33}\sigma_3^2\right).$$

The problem of robust optimal design becomes

$$P_{\mathrm{strip}} = \min_{C\in G_m\mathrm{closure}} \max_{f\in F} \Pi(S,\sigma), \tag{37}$$

where G_m closure is the set of all possible effective compliance tensors of a microstructure formed from the two given materials with the compliance tensors S_A and S_B, taken in the proportion m_A and $m_B = 1 - m_A$, respectively, see [8,28]. We reformulate the problem using a sum of weighted energies formulation, where the minimized functional is taken as a sum of the energies due to the extreme loadings.

6.2. Laminates of the third rank

The description of the strongest structures [17], that minimize the sum of the energies due to several loadings, is known, (see the original papers [2,3,20] and the books [8,30]); the best structures in 2D are so-called "laminates of the third rank" shown in Fig. 5. In 3D, they are the sixth rank laminates [20]. Structural optimization based on using the third rank composites was effectively developed for multi-loadings case in [6,10,26]. The effective compliance tensor $S = C^{-1}$ of a third rank composite – the symmetric fourth-order tensor of elasticity – has the representation

$$S = S_A + m_B((S_B - S_A)^{-1} + m_A N)^{-1}, \tag{38}$$

where S_A is the compliance of an enveloping (reinforcing) material, S_B is the compliance of the material in the nucleus, N is the matrix of structural parameters that depends on the structure of the composite, see [8,30],

$$N = E_A \sum_{i=1}^{3} \alpha_i P(\phi_i), \quad \sum_{i=1}^{3}\alpha_i = 1, \quad \alpha_i \geqslant 0.$$

Here E_A is the Young's modulus of the A-material, angles ϕ_i are the angles that define the directions of laminates (directions of reinforcement), P is a tensor product of four directional vectors $z_i = (\cos\phi_i, \sin\phi_i)$:

$$P(\phi_i) = z_i \otimes z_i \otimes z_i \otimes z_i. \tag{39}$$

α_i are corresponding relative thicknesses of the reinforcing layers in the ith direction.

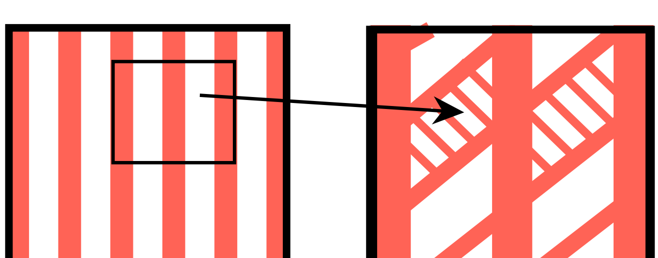

Fig. 5. The schematic picture of the composite of the third rank.

The mentioned symmetry of an optimal composite requires the orthotropy of the optimal structure. Since the original materials are isotropic, the structure is orthotropic if the matrix N is orthotropic. This can be achieved by setting

$$\phi_2 = -\phi_3 = \phi, \quad \alpha_2 = \alpha_3 = \alpha.$$

Generally, the optimal strip is reinforced by three layers of strong material; one layer (with relative volume fraction $1-2\alpha$) is directed in y-direction and two other layers (with equal relative volume fractions α) are symmetrically inclined to the angles $\pm\phi$. In addition, the structure may degenerate into a single layer (when $\alpha = 0$) or two symmetric layers (when $\alpha = \frac{1}{2}$) with angles ϕ and $-\phi$. Because of this symmetry, matrix N for an optimal composite becomes

$$N = (1-2\alpha)P(0) + \alpha P(\phi) + \alpha P(-\phi). \tag{40}$$

Having calculated the compliance of third rank composite, we formulate the structural optimization problem (37) as an algebraic problem

$$J_{\mathrm{strip}} = \min_{\phi,\alpha} \max_{\theta} \Pi_\sigma(S(\phi,\alpha), \sigma(\theta)). \tag{41}$$

We notice that though in general case of minimization of sum of energies corresponding to multiple loadings, the third-rank laminates are optimal, here the optimal structures are the second- not the third-rank laminates (see [12]). Also, the symmetry in this example efficiently reduces the dimension of the computational problem, but the general method works with or without symmetry.

6.3. Numerical example

For the first example, the following values of parameters were chosen

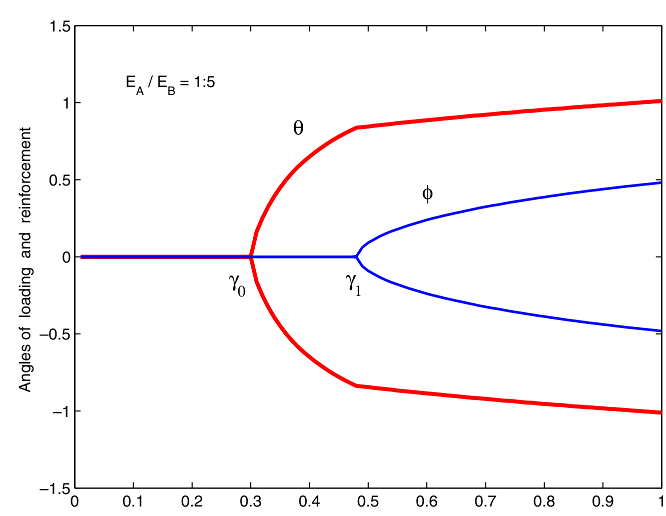

Fig. 6. Bifurcation diagram shows the angle of deviation $\hat{\theta}(\gamma)$ of the extreme, most dangerous loading and the angle $\hat{\phi}(\gamma)$ of optimal reinforcement of the second-rank laminated composite. Notice that the bifurcation parameter γ has different critical values for deviation of the loading θ and for the angle of reinforcement ϕ.

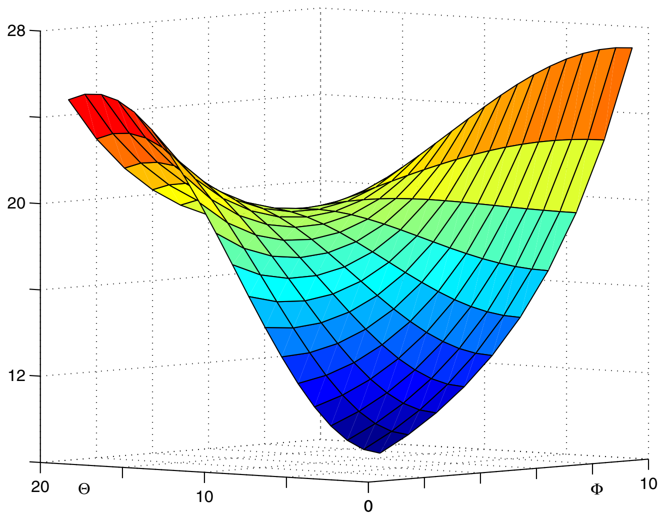

Fig. 7. Energy stored in the composite is a saddle-point function of the angle of deviation of the loading θ and of the direction of reinforcement ϕ.

$$m_A = 1 - m_B = 0.2, \quad E_A = 1, \quad E_B = 5,$$
$$\nu_1 = \nu_2 = 0.3, \quad f_0 = 1.$$

The relative magnitude γ of the variable part of the loading is the parameter of the problem; the angle θ of the optimal deviation of the extreme loading and the structural parameters α and ϕ are determined from the solution of the min–max optimization problem. We find three regimes:

1. When $\gamma < \gamma_0 = .31$, the extreme loading is vertical, $\theta_{\text{opt}} = 0$, and the optimal structure is a laminate with vertical layers directed across the strip, $\phi_{\text{opt}} = 0$, see Fig. 6.
2. At the critical value γ_0 of the parameter γ, the direction of the extreme deviation undergoes a bifurcation, $\theta_{\text{opt}} = \pm\hat{\theta}(\gamma)$. But while $\gamma < \gamma_1 = .46$, the optimal structure remains the same laminate with layers directed across the strip, $\phi_{\text{opt}} = 0$ (Fig. 6).
3. When the magnitude γ further increases, $\gamma \geqslant \gamma_1$, the optimal structure bifurcates as well; it becomes a second-rank matrix laminate with the angle of reinforcement $\phi_{\text{opt}} = \pm\hat{\phi}(\gamma)$ (curve marked ϕ in Fig. 6).

Although the problem has two solutions for the extreme loading, due to the symmetry, the dependence of the compliance on the parameters ϕ and θ is a saddle-point surface, it is shown in Fig. 7. Indeed, the problem is reformulated (relaxed) accounting for non-uniqueness of the loading and for the symmetry in the design.

Fig. 8 summarizes the dependence of the optimal solution on the ratio of Young's module of the materials in the composite. It shows bifurcation diagrams for different ratios of the Young's moduli. Qualitatively, the picture remains the same, but the critical values of the bifurcation parameter γ are different: The larger is the ratio, the smaller the critical value of γ_0 and γ_1 at which the bifurcation occurs. An interval (γ_0, γ_1) decreases with the increase of the ratio of Young's moduli.

7. Conclusions

The principal compliance is a basic characteristic of an elastic body which depends only on the shape of the domain and on the stiffness of the material. By the proper normalization of Λ by $\|\Omega\|$ and $\|C\|$, this quantity is reduced to dimensionless parameter λ:

$$\lambda = \frac{\Lambda}{||\Omega||||C||},$$

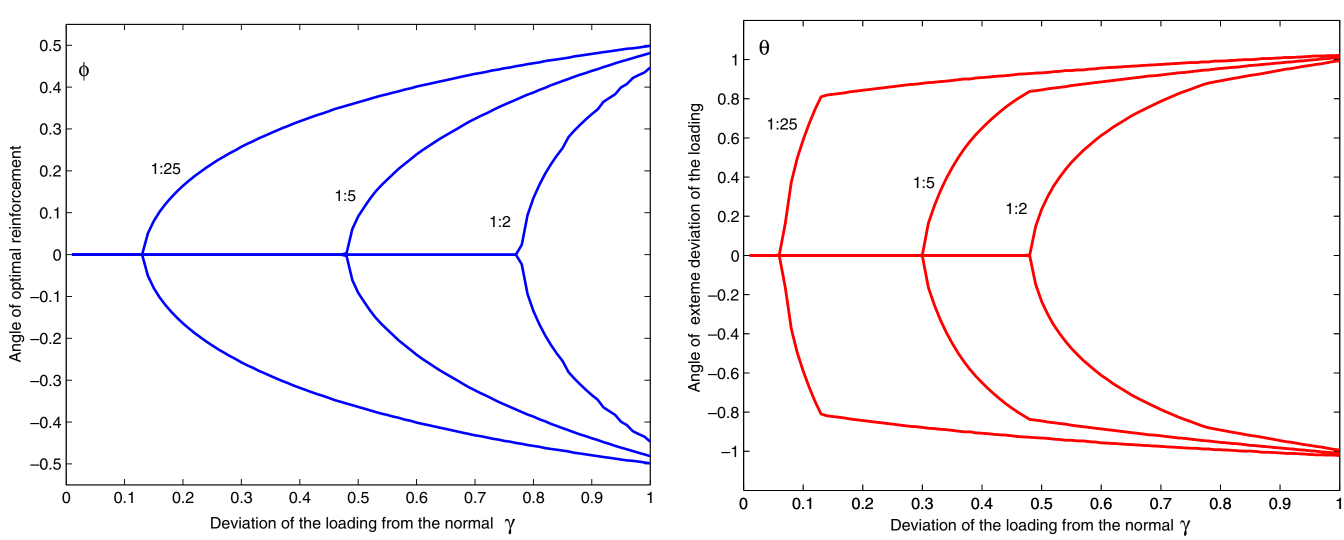

Fig. 8. Pitchfork bifurcation diagram for different ratios of Young's moduli of materials in the composite ranging from 1:2 to 1:25. Left: Bifurcation of the angle $\hat{\phi}(\gamma)$ of direction of the optimal reinforcement for the second-rank laminated composite. Right: Bifurcation of the angle $\hat{\theta}(\gamma)$ of the deviation of the extreme loading from the normal.

and can be treated as a basic integral characteristic of the filled domain alongside with such properties as the principal eigenfrequency, capacity, etc. Notice, that λ depends on the class of admissible loadings. Therefore, it is able to provide various characteristics of the domain.

The optimal design aimed to decrease the principal compliance is found by solving a minimax problem; typically, the problem does not have a saddle point and the optimal design provides equal minimal compliance for several extreme loadings. Symmetries and relaxation bring the problem to a saddle-point type. Depending on the type of constraints, the extreme loading can be a principal eigenfunction of an eigenvalue problem, a concentrated loading, or a solution of a bifurcation problem.

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