Happy numbers

Given a number n let $f(n)$ be the sum of the squares of its digits (in base 10). For example $f(26) = 2^2 + 6^2 = 40$. A number n is called *happy* if there exists a k so that $\underbrace{f \circ f \circ ... \circ f}_{k \text{ times}}(n) = 1$. For example, 7 is a happy number because $\underbrace{f \circ f \circ ... \circ f}_{5 \text{ times}}(7) = 1$.

Getting started:

(1) Show that 7 is a happy number.
(2) Show that there are infinitely many happy numbers.
(3) Show that 2 is not a happy number.
(4) Show that there are infinitely many numbers that are not happy.

Getting a better feel:

(1) Find a number k so that $f(n) < n$ for all $n > k$. Show that if $f(n) = n$ then $n < 100$.
(2) A sequence of numbers $a_1, a_2, ...$ is called *eventually periodic* if there exists d and p so that $a_{i+p} = a_i$ whenever $i \geq d$. So if we look at the sequence of decimal digits of $\frac{1}{6} = .166666...$ we get the eventually periodic sequence $1, 6, 6, 6, ...$. In this case d can be chosen to be 2 and p can be chosen to be 1.

    Show that if n is not happy then the sequence $f(n), f \circ f(n), f \circ f \circ f(n), ...$ is eventually periodic. For an example of the sequence I am talking about, if $n = 2$ then the sequence $f(2), f \circ f(2), f \circ f \circ f(2), f \circ f \circ f \circ f(2), ...$ is $2, 4, 16, 37, ...$
(3) Let's call $f(i), f(i+1), ..., f(i+p-1)$ as above the *terminating cycle* and p the *eventual period*. Assume that n is not happy and consider the sequence $f(n), f \circ f(n), f \circ f \circ f(n), ...$ Can you come up with an upper bound for the eventual period p of this sequence?

(4) Show that if $f(n) = n$ then $n = 1$.
(5) Show that if n is not happy then $f(n), f \circ f(n), ...$ has a terminating cycle $4, 16, 37, 73, 58, 89, 145, 42, 20$.

Going further At the start of this worksheet, I mentioned "base 10". When we write the string 134 we are saying it is

$$1 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0.$$

We are writing it as a sum of numbers in $\{0, 1, ..., 8, 9\}$ times powers of 10. We can write numbers in other bases as well. For example, if we want to write a number in base 5 we want to represent it as a sum of numbers in $\{0, 1, 2, 3, 4\}$ times powers of 5. The number that the string 134 represents in base 5 is

$$1 \cdot 5^2 + 3 \cdot 5^1 + 4 \cdot 5^0.$$

More generally the string $d_k d_{k-1}...d_1 d_0$ in base m represents the number

$$d_k m^k + d_{k-1} m^{k-1} + ... + d_1 m^1 + d_0 m^0.$$

We request that the numbers $d_0, d_1, ...$ are all in $\{0, 1, ..., m-2, m-1\}$. If $n = d_k d_{k-1}...d_1 d_0$ in base m, that is

$$n = d_k \cdot m^k + d_{k-1} \cdot m^{k-1} + ... + d_1 \cdot m + d_0 \cdot m^0,$$

define $f_m(n)$ to be $d_0^2 + d_1^2 + ... + d_{k-1}^2 + d_k^2$. So $f_5(1 \cdot 5^2 + 3 \cdot 5 + 4^2) = 2 \cdot 10 + 6 = 1 \cdot 5^2 + 0 \cdot 5 + 1$.

(1) Find a number m and an $n > 1$ so that $f_m(n) = n$.
(2) Show that if $f_m(n) = n$ then n is a 3 digit number in base m. That is $n < m^3$. ($m^3 = 1 \cdot m^3 + 0 \cdot m^2 + 0 \cdot m + 0$ is the smallest four digit number in base m.)
(3) Can you find infinitely many m so that there exists $n > 1$ with $f_m(n) = n$?
(4) Is there any number n that has 3 digits in base m and so that $f_m(n) = n$?

A closing remark: Happy numbers have the following remarkable property: There are infinitely many numbers n so that the number of happy numbers less than n is greater than $.18 \cdot n$ and there are infinitely many numbers n so that the number of happy numbers less than n is less than $.12 \cdot n$. In jargon, the set of happy numbers doesn't have density. To find out more about this, see Justin Glimers "On the density of happy numbers," which was published in *Integers* in 2013. It is also available at: https://arxiv.org/pdf/1110.3836.pdf