Lebesgue Measure

An n-*cell* is a product of n-intervals in $\mathbb{R}^n$. We allow the intervals to be of any type; open, closed or half open. We also allow an interval to be a single point. That is $[a, a]$ is an allowed interval. If $Q = I_1 \times \cdots \times I_n$ is an n-cell and the endpoints of the interval I_i are $a_i < b_i$ then define the *measure* of Q by

$$m(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

Problem 1 Let Q be an n-cell and let $Q_1, \dots, Q_k$ be disjoint n-cells with $Q = \bigcup_{i=1}^k Q_i$. Show that $m(Q) = \sum_{i=1}^k m(Q_i)$.

A subset of $\mathbb{R}^n$ is *elementary* if its a finite union of n-cells. Let $\mathcal{E}$ be the set of all elementary subsets.

Problem 2 If $A, B \in \mathcal{E}$ show that $A \cup B$, $A \cap B$ and $A - B$ are all in $\mathcal{E}$.

Problem 3 Show that an elementary set is a finite union of disjoint n-cells.

We can now define the measure of an elementary set. If $A \in \mathcal{E}$ let $Q_1, \dots, Q_k$ be disjoint n-cells with $A = \bigcup_{i=1}^k Q_i$. Then

$$m(A) = \sum_{i=1}^k m(Q_i).$$

Problem 4 Show that the measure of an elementary set is well defined. That is if $A \in \mathcal{E}$ and $Q_1, \dots, Q_k$ and $R_1, \dots, R_\ell$ are two collections of disjoint n-cells with $A = \bigcup_{i=1}^k Q_i = \bigcup_{i=1}^\ell R_i$ then

$$\sum_{i=1}^k m(Q_i) = \sum_{i=1}^\ell m(R_i).$$

Problem 5 Let A and B be elementary sets. Show that

$$m(A) + m(B) = m(A \cup B) + m(A \cap B).$$

Proposition 1 *Given any $A \in \mathcal{E}$ and any $\epsilon > 0$ there exists an open $U \in \mathcal{E}$ and a closed $F \in \mathcal{E}$ with $F \subset A \subset U$ and $m(U) - \epsilon \leq m(A) \leq m(F) + \epsilon$.*

Proof. Let $Q_1, \dots, Q_k$ be a disjoint collection of n-cells such that $A = \bigcup Q_i$. For each Q_i we can find an open n-cell, U_i and a closed n-cell, F_i, such that $F_i \subset Q_i \subset U_i$ and $m(U_i) - \epsilon/k \leq m(Q_i) \leq m(F_i) - \epsilon/k$. Then $U = \bigcup U_i$ and $F = \bigcup F_i$ are the desired U and F. $\boxed{1}$

We can now define the measure of an arbitrary subset of $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ define

$$m^*(A) = \inf_{\substack{A \subset \bigcup_{i=1}^\infty A_i \\ A_i \in \mathcal{E} \text{ is open}}} \sum_{i=1}^\infty m(A_i).$$

If A is an elementary set then this gives a new way of defining the measure of A. We need to see that both ways give the same answer.

Proposition 2 *If $A \in \mathcal{E}$ then $m^*(A) = m(A)$.*

Proof. By Proposition 1 for all $\epsilon > 0$ there is an open $U \in \mathcal{E}$ such that $A \subset U$ and $m(A) \geq m(U) - \epsilon$. Then $m(U) \geq m^*(A)$ so $m(A) \geq m^*(A) - \epsilon$. As this is true for arbitrary $\epsilon > 0$ we must have $m(A) \geq m^*(A)$.

Applying Proposition 1 again we have a closed n-cell F with $F \subset A$ and $m(A) \leq m(F) + \epsilon/2$. By the definition of $m^*(A)$ we can also find a countable collection of open elementary sets A_i such that $A \subset \bigcup A_i$ and $m^*(A) \geq \sum m(A_i) - \epsilon/2$. The set F is closed and bounded so it is compact. It is also covered by the open set A_i so there is a finite subset of the A_i with $F \subset \bigcup_{i=1}^k A_k$ and therefore

$$m(F) \leq m\left(\bigcup_{i=1}^k A_i\right) \leq \sum_{i=1}^k m(A_i) \leq \sum_{i=1}^\infty m(A_i).$$

Combing the inequalities we have

$$m(A) \leq m(F) + \epsilon/2 \leq \sum_{i=1}^\infty m(A_i) + \epsilon/2 \leq m^*(A) + \epsilon.$$

As this holds for all $\epsilon > 0$ we have $m(A) \leq m^*(A)$ finishing the proof. $\boxed{2}$

Proposition 3 *If $A = \bigcup_{i=1}^\infty A_i$ then $m(A) \leq \sum_{i=1}^\infty m(A_i)$.*

Proof. We can assume the sum is finite for otherwise the proposition is obviously true. Fix an $\epsilon > 0$. Let A_{ij} be open elementary sets such that $A_i \subset \bigcup_{j=1}^\infty A_{ij}$ and $m(A_i) \geq \sum_{j=1}^\infty m(A_{ij}) - \epsilon/2^i$. Since

$$A \subset \bigcup_{i=1}^\infty A_i \subset \bigcup_{i=1}^\infty \left(\bigcup_{j=1}^\infty A_{ij}\right)$$

and there are a countable number of A_{ij} we have

$$m(A) \le \sum_{i,j=1}^{\infty} m(A_{ij}) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} m(A_{ij})\right) \le \sum_{i=1}^{\infty} m(A_i) + \sum_{i=1}^{\infty} \epsilon/2^i = \sum_{i=1}^{\infty} m(A_i) + \epsilon.$$

Since ϵ was arbitrary we have $m(A) \le \sum_{i=1}^{\infty} m(A_i)$ as desired. $\boxed{3}$

Define the *symmetric difference* of two sets A and B by

$$S(A, B) = (A - B) \cup (B - A).$$

We then define a distance between two sets by

$$d(A, B) = m(S(A, B)).$$

We can use $d(,)$ to define a notion of convergence for sets. Namely $A_i \to A$ if the $d(A_i, A) \to 0$ as $i \to \infty$. Note that, unlike convergence for points, it can happen that $A_i \to A$ and $A_i \to A'$ but $A \neq A'$.

Problem 6 Show that

$$\begin{array}{l} S(A_1 \cup A_2, B_1 \cup B_2) \\ S(A_1 \cap A_2, B_1 \cap B_2) \quad \subset S(A_1, B_1) \cup S(A_2, B_2) \\ S(A_1 - A_2, B_1 - B_2) \end{array}$$

and that

$$\begin{array}{l} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \quad \le d(A_1, B_1) + d(A_2, B_2). \\ d(A_1 - A_2, B_1 - B_2) \end{array}$$

Also show that

$$S(A, C) \subset S(A, B) \cup S(B, C)$$

and that

$$d(A, C) \le d(A, B) + d(B, C).$$

Lemma 4 *1. For any two sets A, B we have $|m(A) - m(B)| \le d(A, B)$.*

2. If $A_i \to A$ then $m(A_i) \to m(A)$.

Proof of (a). Without loss of generality we assume $m(A) \geq m(B)$. By the previous problem

$$d(A, \emptyset) \leq d(A, B) + d(B, \emptyset).$$

Since $d(X, \emptyset) = m(X)$ for any set X and $m(A) \geq m(B)$ we have

$$0 \leq m(A) - m(B) \leq d(A, B)$$

or $|m(A) - m(B)| \leq d(A, B)$.

(b). By (a), $|m(A) - m(A_i)| \leq d(A, A_i)$. Since $d(A, A_i) \to 0$ we have $m(A_i) \to m(A)$. $\boxed{4}$

Define

$$\mathfrak{M}_F = \{A \subset \mathbb{R}^n | \text{there exists } A_i \in \mathcal{E} \text{ with } A_i \to A\}$$

and

$$\mathfrak{M} = \left\{ A \subset \mathbb{R}^n | \text{there exists } A_i \in \mathfrak{M}_F \text{ with } A = \bigcup_{i=1}^{\infty} A_i \right\}.$$

Proposition 5 *Let A, B be in $\mathfrak{M}_F$. Then*

1. $A \cup B \in \mathfrak{M}_F$;
2. $A \cap B \in \mathfrak{M}_F$;
3. $A - B \in M_F$;
4. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.

Proof of (a)-(c). Since A and B are in $\mathfrak{M}_F$ we have A_i and B_i in $\mathcal{E}$ with $A_i \to A$ and $B_i \to B$. By Problem 2, $A_i \cup B_i$, $A_i \cap B_i$ and $A_i - B_i$ are all in $\mathcal{E}$. An application of Problem 6 implies that $A_i \cup B_i \to A \cup B$, $A_i \cap B_i \to A \cap B$ and $A_i - B_i \to A - B$.

(d). Since the A_i and B_i are in $\mathcal{E}$ we have

$$m(A_i) + m(B_i) = m(A_i \cup B_i) + m(A_i \cap B_i)$$

by Problem 5. As $i \to \infty$ this becomes

$$m(A) + m(B) = m(A \cup B) + m(A \cap B)$$

by (b) of Lemma 4. $\boxed{5}$

Lemma 6 *Let $A = \bigcup_{i=1}^{\infty} A_i$ where the A_i are pairwise disjoint and in $\mathfrak{M}_F$. Then*

$$m(A) = \sum_{i=1}^{\infty} m(A_i).$$

Proof. We have already seen that $m(A) \leq \sum_{i=1}^{\infty} m(A_i)$ so we only need to reverse the inequality. By (1) of Lemma 5 the finite union $B_k = \bigcup_{i=1}^{k} A_i$ is in $\mathfrak{M}_F$ since the A_i are in $\mathfrak{M}_F$. Next we apply (4) of Lemma 5 to see that $m(B_k) = \sum_{i=1}^{k} m(A_i)$. Since $B_k \subset A$ we have $\sum_{i=1}^{k} m(A_i) = m(B_k) \leq m(A)$. As all the terms in the series $\sum_{i=1}^{\infty} m(A_i)$ are non-negative and its partial sums are bounded above by $m(A)$ we have that the series converges and $\sum_{i=1}^{\infty} m(A_i) \leq m(A)$. $\boxed{6}$

Lemma 7 *Let A be a set in $\mathfrak{M}$. Then $m(A) < \infty$ if and only if $A \in \mathfrak{M}_F$.*

Proof. If $A \in \mathfrak{M}_F$ then there exists a $B \in \mathcal{E}$ such that $d(A, B) < \infty$. Since $A \subset B \cup S(A, B)$ and $m(B) < \infty$ this implies that $m(A) < \infty$.

Now assume that $m(A) < \infty$. Since $A \in \mathfrak{M}$ there are $A_i \in \mathfrak{M}_F$ with $A = \bigcup_{i=1}^{\infty} A_i$. Let $B_k = \bigcup_{i=1}^{k} A_i$. For $i > 1$ we define $A'_i = B_i - B_{i-1}$ and for $i = 1$ we define $A'_1 = A_1$. Applying (1) and (3) of Lemma 5 we see that the B_i and A'_i are in $\mathfrak{M}_F$. The A'_i are disjoint and their union is A so $m(A) = \sum_{i=1}^{\infty} m(A'_i)$ by Lemma 6. We also have $B_k \to A$ since

$$d(A, B_k) = m(S(A, B_k)) = m\left(\bigcup_{i=k+1}^{\infty} A'_i\right) = \sum_{i=k+1}^{\infty} m(A'_i)$$

and the sum on the right limits to 0 as $k \to \infty$ since it is the tail of a convergent series. Note that the last equality is an application of Lemma 6.

Each B_i is in $\mathfrak{M}_F$ so we can find $E_i \in \mathcal{E}$ with $d(B_i, E_i) \leq 2^{-i}$. Then

$$d(A, E_i) \leq d(A, B_i) + d(B_i, E_i) \leq d(A, B_i) + 2^{-i}.$$

The right side of the inequality limits to 0 since $B_i \to A$ and $2^{-i} \to 0$ and therefore $E_i \to A$ and A is in $\mathfrak{M}_F$. $\boxed{6}$

Theorem 8 *If $A \in \mathfrak{M}$ and $A_i \in \mathfrak{M}$ with the A_i disjoint and $A = \bigcup_{i=1}^{\infty} A_i$ then*

$$m(A) = \sum_{i=1}^{\infty} m(A_i).$$

Proof. If $m(A_i) < \infty$ for all i then by Lemma 7, A_i is in $\mathfrak{M}_F$ for all i and the theorem follows from Lemma 6. If $m(A_i) = \infty$ for some i then $m(A) \geq m(A_i)$ is also infinite and again the equality holds. $\boxed{8}$

Theorem 9 *1. If $A_i \in \mathfrak{M}$ then $\bigcup_{i=1}^{\infty} A_i$ is in $\mathfrak{M}$.*

2. If A and B are in $\mathfrak{M}$ then $A \cap B$ is in $\mathfrak{M}$.

3. If A and B are in $\mathfrak{M}$ then $A - B$ is in $\mathfrak{M}$.

Proof of (1). Since A_i is in $\mathfrak{M}$ there are $A_{ij} \in \mathfrak{M}_F$ with $A_i = \bigcup_{j=1}^{\infty} A_{ij}$. Since a countable collection of countable sets is countable there are countably many A_{ij}. Therefore $A = \bigcup_{i,j=1}^{\infty} A_{ij}$ is in $\mathfrak{M}$.

(2) and (3). Let A_i and B_i be countable collections of sets in $\mathfrak{M}_F$ whose unions are A and B, respectively. Then each $A_i \cap B_j$ is in $\mathfrak{M}_F$ by Lemma 5 and by (1) we have that

$$A_i \cap B = \bigcup_{j=1}^{\infty} A_i \cap B_j$$

is in $\mathfrak{M}$. Another application of (1) then gives us that

$$A \cap B = \bigcup_{i=1}^{\infty} A_i \cap B$$

is in $\mathfrak{M}$.

For (3) we note that

$$A - B = \bigcup_{i=1}^{\infty} A_i - B.$$

Since $A_i - B = A_i - (A_i \cap B)$ is in $\mathfrak{M}$ the union is also in $\mathfrak{M}$. $\boxed{9}$