**Notes and problems on the Lebesgue integral**

Let $X \in \mathfrak{M}$ be a measurable subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is *measurable* if $f^{-1}([a, \infty))$ is a measurable for all $a \in \mathbb{R}$.

**Problem 1** Prove that the following are equivalent:

1. $f^{-1}([a, \infty))$ is measurable for all $a \in \mathbb{R}$;
2. $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$;
3. $f^{-1}((\infty, a])$ is measurable for all $a \in \mathbb{R}$;
4. $f^{-1}((\infty, a))$ is measurable for all $a \in \mathbb{R}$;
5. $f^{-1}((a, b))$ is measurable for all $a < b$ in $\mathbb{R}$;
6. $f^{-1}(U)$ is measurable for all open subsets of $\mathbb{R}$.

**Problem 2** Let $f$ and $g$ be measurable functions. Show that the set

$$\{x \in X | f(x) \geq g(x)\}$$

is measurable.

**Theorem 1** *Let $f_i$ be a countable collection of measurable functions. Then*

1. $\inf f_i$ *is measurable;*
2. $\sup f_i$ *is measurable;*
3. $\liminf f_i$ *is measurable;*
4. $\limsup f_i$ *is measurable.*

**Proof.** Let $h = \inf f_i$. Then $h^{-1}((\infty, a]) = \bigcup f_i^{-1}((\infty, a]))$ is a countable of union of measurable sets and is therefore measurable. This implies that $h$ is measurable by (3) of Problem 1. A similar argument for $\sup f_i$ where replace the sets $(\infty, a]$ with sets $[a, \infty)$.

For (3) if we let $g_k = \inf\limits_{i \geq k} f_i$ then $g = \sup g_k = \liminf f_i$. All of the $g_k$ are measurable by (1) so $g$ is measurable by (2). Reversing the roles of sup and inf we also get (4). $\boxed{1}$

**Corollary 2** *1. A limit of measurable functions is measurable.*

*2. If $f$ and $g$ are measurable then* $\min\{f, g\}$ *and* $\max\{f, g\}$ *are measurable.*

**Proof.** The limit of $f_i$ exists if and only if $\limsup f_i = \liminf f_i$ so (1) follows from (3) or (4) of Theorem 1.

(2) is a special case of (1) and (2) of Theorem 1. $\boxed{2}$

**Theorem 3** *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and let $u, v : X \to \mathbb{R}$ be measurable. Then the function $f(x) = \Phi(u(x), v(x))$ is measurable.*

**Proof.** Let $h(x) = (u(x), v(x))$ and let $R = I_1 \times I_2$ be a rectangle in $\mathbb{R}^2$. Then $h^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2)$ is measurable since $u^{-1}(I_1)$ and $v^{-1}(I_2)$ are measurable. Since every open set in $\mathbb{R}^2$ is a countable union of rectangles this implies that $h^{-1}(U)$ is measurable for any open set $U$.

To see that $f$ is measurable we observe that for any open interval $(a, b)$ we have $f^{-1}((a, b)) = h^{-1}(\Phi^{-1}(a, b))$ is measurable since $\Phi^{-1}((a, b))$ is open by the continuity of $\Phi$ and $h^{-1}$ of any open set is measurable. $\boxed{3}$

**Corollary 4** *If $u, v : X \to \mathbb{R}$ are measurable then $u + v$ and $u \cdot v$ are measurable.*

**Proof.** Apply Theorem 3 to the functions $(x, y) \mapsto x + y$ and $(x, y) \mapsto x \cdot y$. $\boxed{4}$

For $E \subset X$ we define the *characteristic function* by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

**Problem 3** Show that $\chi_E$ is measurable if and only if $E \in \mathfrak{M}$.

A *simple function* is a finite linear combination of characteristic functions. That is $s = \sum_{i=1}^k c_i \chi_{E_i}$ where $c_i \in \mathbb{R}$. Corollary 4 and Problem 3 implies that if all the $E_i$ are in $\mathfrak{M}$ then $s$ is measurable.

**Problem 4**

1. Let $s = \sum_{i=1}^k c_i \chi_{E_i}$ be a simple, measurable function and assume that all of the $c_i$ are distinct ($c_i = c_j$ if and only $i = j$). Show that all of the $E_i$ are measurable.
2. Show that every simple measurable function can be written as a finite linear combination of characteristic functions of disjoint measurable sets.

For a simple function we define the *Lebesgue integral* by

$$\int_X s(x)dm(x) = \sum_{i=1}^{k} c_i m(E_i \cap X).$$

**Lemma 5** *Let $s$ and $r$ be simple, meaurable functions, $c$ a real number and $A$ and $B$ disjoint measurable sets. Then*

1. $$\int_A cs(x)dm(x) = c\int_A s(x)dm(x);$$

2. $$\int_A s(x)dm(x) + \int_A r(x)dm(x) = \int_A (s+r)(x)dm(x);$$

3. $$\int_A s(x)dm(x) + \int_B s(x)dm(x) = \int_{A\cup B} s(x)dm(x).$$

**Proof.** We'll only prove (3) and the leave (1) and (2) to the reader. Let $s = \sum_{i=1}^{k} c_i \chi_{E_i}$ where the $E_i$ are measurable sets. Then

$$\begin{aligned}
\int_A s(x)dm(x) + \int_B s(x)dm(x) &= \sum_{i=1}^{k} c_i m(E_i \cap A) + \sum_{i=1}^{k} c_i m(E_i \cap B) \\
&= \sum_{i=1}^{k} c_i \left(m(E_i \cap A) + m(E_i \cap B)\right) \\
&= \sum_{i=1}^{k} c_i m(E_i \cap (A \cup B)) \\
&= \int_{A\cup B} s(x)dm(x).
\end{aligned}$$

$\boxed{5}$

Let $f : X \to \mathbb{R}$ be a non-negative function and let $\mathcal{S}(f)$ be the set of simple, measurable functions with $s \geq 0$ and $s \leq f$. Then we define the *Lebesgue integral* of $f$ by

$$\int_X f(x)dm(x) = \sup_{s\in\mathcal{S}(f)} \int_X s(x)dm(x).$$

**Theorem 6** *Let $f : X \to \mathbb{R}$ be a measurable, non-negative function. Then there exist simple, measurable functions $0 \leq s_1 \leq s_2 \leq \ldots$ such that $s_i(x) \to f(x)$ for all $x \in X$.*

**Proof.** We define each $s_i$ as follows. Define

$$E_j = f^{-1}([j2^{-1}, (j+1)2^{-i})$$

for $j = 0, \ldots, i2^i - 1$ and define

$$E_{i2^i} = f^{-1}([i, \infty)).$$

Since $f$ is measurable all of the $E_j$ are measurable and

$$s_i = \sum_{j=0}^{i2^i} j2^{-i}\chi_{E_j}.$$

Note that all the $E_j$ are disjoint and their union is $X$. Therefore if $x \in E_j$, $s_i(x) = j2^{-i} \leq f(x)$. Also if $f(x) < i$ we $f(x) - s_i(x) < 2^{-1}$ so $s_i(x) \to f(x)$ as $i \to \infty$. $\boxed{4}$

**Lemma 7** *For each $E \in \mathfrak{M}$ define a function $\phi_E : \mathfrak{M} \to [0, \infty]$ by*

$$\phi_E(A) = \int_A \chi_E(x)dm(x).$$

*If $A_i \in \mathfrak{M}$ are disjoint and $A = \bigcup_{i=1}^{\infty} A_i$ then*

$$\phi_E(A) = \sum_{i=1}^{\infty} \phi_E(A_i).$$

**Proof.** For any $B \in M$ we have $\phi_E(B) = m(E \cap B)$. Since $A_i \cap E$ are disjoint sets in $\mathfrak{M}$ and $A \cap E = \bigcup_{i=1}^{\infty} A_i \cap E$ we have

$$\phi_E(A) = m(E \cap A) = \sum_{i=1}^{\infty} m(E \cap A_i) = \sum_{i=1}^{\infty} \phi_E(A_i)$$

by the countable additivity of the measure $m$. $\boxed{7}$

**Proposition 8** *Let $s$ be a non-negative, simple, measurable function and define $\phi_s : \mathfrak{M} \to [0, \infty]$ by*

$$\phi_s(A) = \int_A s(x) dm(x).$$

*If $A_i \in \mathfrak{M}$ are disjoint and $A = \bigcup_{i=1}^{\infty} A_i$ then*

$$\phi_s(A) = \sum_{i=1}^{\infty} \phi_s(A_i).$$

**Proof.** By (2) of Problem 4 there are disjoint $E_i \in \mathfrak{M}$ with $s = \sum_{i=1}^{k} c_i \chi_{E_i}$. Since $s$ is non-negative the $c_i$ are also non-negative. We can even assume they are positive since if $c_i = 0$ we can clearly drop that term from the sum. Then for any set $B$ we have $\phi_s(B) = \sum_{i=1}^{k} c_i \phi_{E_i}(B)$. Therefore if $\phi_s(A) < \infty$ or $\sum_{i=1}^{\infty} \phi_s(A_i) < \infty$ we have

$$\phi_s(A) = \sum_{i=1}^{k} c_i \phi_{E_i}(A) = \sum_{i=1}^{k} c_i \left( \sum_{j=1}^{\infty} \phi_{E_i}(A_j) \right) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{k} c_i \phi_{E_i}(A_j) \right) = \sum_{j=1}^{\infty} \phi_s(A_j).$$

Note that we are allowed to switch the order of summation (the 3rd equality) since the total sum is finite and all terms are positive. In particular if one side of the equality is finite so is the other side. $\boxed{8}$

**Corollary 9** *Let $A_1 \subset A_2 \dots$ be nested measurable sets with $A = \bigcup_{i=1}^{\infty} A_i$ and let $s$ be a measurable simple function. Then*

$$\lim_{i \to \infty} \int_{A_i} s(x) dm(x) = \int_A s(x) dm(x).$$

**Proof.** Let $B_1 = A_1$ and let $B_i = A_i - A_{i-1}$ for $i \geq 2$. Note that the $B_i$ are disjoint and measurable, $A = \bigcup_{i=1}^{\infty} B_i$ and $A_k = \bigcup_{i=1}^{k} A_i$. Then

$$\begin{aligned}
\int_A s(x)dm(x) &= \phi_s(A) \\
&= \sum_{i=1}^{\infty} \phi_s(B_i) \text{ by Proposition 8} \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \phi_s(B_i) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \int_{B_i} s(x)dm(x) \\
&= \lim_{k\to\infty} \int_{A_k} s(x)dm(x).
\end{aligned}$$

$\boxed{9}$

**Theorem 10 (Lebesgue's Monotone Convergence Theorem)** *Let $f_i \geq 0$ be a sequence of non-negative measurable functions. Assume that $f_i \leq f_{i+1}$ and $f_i \to f$, pointwise, for some function $f$. The $f$ is measurable and*

$$\lim_{i\to\infty} \int_X f_i(x)dm(x) = \int_X f(x)dm(x).$$

**Proof.** By (1) of Corollary 2 the function $f$ is measurable.

Since the $f_i$ are increasing we have $f_i \leq f_{i+1} \leq f$ and

$$\int_X f_i dm(x) \leq \int_X f_{i+1} dm(x) \leq \int_X f(x)dm(x).$$

Therefore $\int_X f_i(x)dm(x)$ is an increasing sequence so it must converge to some $\lambda \in [0,\infty]$ with $\lambda \leq \int_X f(x)dm(x)$. We need to prove the reverse inequality.

For all $\epsilon > 0$ we can find a simple, measurable function $g$ with $0 \geq g \leq f$ and

$$\int_X f(x)dm(x) - \int_X g(x)dm(x) < \epsilon.$$

For $c \in (0,1)$ define

$$E_i = \{x \in X | f_i(x) \geq cg(x)\}.$$

By Problem 2, the $E_i$ are measurable. Note that $E_i \subset E_{i+1}$ and every $x \in X$ is contained in some $E_i$ so $E_i$ is a nested sequence with $X = \bigcup_{i=1}^{\infty} E_i$. We then have

$$\int_X f_i(x)dm(x) \geq \int_{E_i} f_i(x)dm(x) \geq c\int_{E_i} g(x)dm(x). \tag{0.1}$$

The limit of the left hand side of this inequality is $\lambda$ and by Corollary 9 the limit of the right hand side is the integral of $g$ over $X$. In particular we have

$$\lambda \geq c\int_X g(x)dm(x) \geq c\left(\int_X f(x)dm(x) - \epsilon\right)$$

for all $c \in (0,1)$ and all $\epsilon > 0$. Therefore

$$\lambda \geq \int_X f(x)dm(x)$$

and the proof is done. $\boxed{10}$

**Theorem 11** *Let $f \geq 0$ and $g \geq 0$ be measurable, non-negative functions, $c$ a real number and $A$ and $B$ disjoint measurable sets. Then*

1. $$\int_A cf(x)dm(x) = c\int_A f(x)dm(x);$$

2. $$\int_A f(x)dm(x) + \int_A g(x)dm(x) = \int_A (f+g)(x)dm(x);$$

3. $$\int_A f(x)dm(x) + \int_B f(x)dm(x) = \int_{A\cup B} f(x)dm(x).$$

**Proof.** By Theorem 6 we can find non-decreasing sequences $s_i$ and $r_i$ of simple, non-negative, measurable functions with $s_i \to f$ and $r_i \to g$. Note that $cs_i$ and $s_i + r_i$ are also non-decreasing sequences of simple, non-negative, measurable functions with $cs_i \to cf$ and $s_i + r_i \to f + g$. If we replace $f$ and $g$ by $s_i$ and $r_i$ then (1)-(3) all hold by Lemma 5. By the Theorem 10 the statements all hold in the limit also. $\boxed{11}$

We now define the Lebesgue integral of an arbitrary function. Given a function $f$ set $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then $f = f^+ - f^-$. Define the integral of $f$ by

$$\int_A f(x)dm(x) = \int_A f^+(x)dm(x) - \int_A f^-(x)dm(x).$$

Note that if $f$ is measurable so are both $f^+$ and $f^-$.

**Theorem 12 (Fatou's Lemma)** *Let $f_i$ be non-negative measurable functions then*

$$\int_A (\liminf_{i\to\infty} f_i)(x)dm(x) \leq \liminf \int_A f_i(x)dm(x).$$

**Proof.** Let $g_k = \inf_{i\geq k} f_i$. Then $g_k \leq f_i$ if $i \geq k$ and therefore

$$\int_A g_k(x)dm(x) \leq \int_A f_i(x)dm(x)$$

and

$$\int_A g_k(x)dm(x) \leq \inf_{i\geq k} \int_A f_i(x)dm(x).$$

Also note that $g_k \leq g_{k+1}$ and $g_k \to \liminf f_i$ so the Monotone Convergence Theorem applies to the left hand side of the inequalities and as $k \to \infty$ we get

$$\int \lim_{k\to\infty} g_k(x)dm(x) \leq \lim_{k\to\infty} \left( \inf_{i\geq k} \int_A f_i(x)dm(x) \right)$$

which is exactly the desired inequality. $\boxed{12}$

**Theorem 13 (Lebesgue's Dominated Convergence Theorem)** *Let $f_i$ be a sequence of measurable functions with $f_i \to f$. Assume that $|f_i| \leq g$ where $g$ is measurable and $\int_A g(x)dm(x) < \infty$. Then*

$$\lim_{i\to\infty} \int_A f_i(x)dm(x) = \int_A f(x)dm(x).$$

**Proof.** Since $|f_i| \leq g$, $-f_i \leq g$ and $0 \leq f_i + g$. By Fatou's Lemma

$$\int_A \liminf(f_i + g)(x)dm(x) \leq \liminf \int_A (f_i + g)(x)dm(x).$$

Note that $\liminf(f_i + g) = f + g$ since $f_i$ converges to $f$ and $g$ doesn't depend on $i$. Therefore the left hand side of the inequality becomes

$$\int_A (f+g)(x)dm(x) = \int_A f(x)dm(x) + \int_A g(x)dm(x).$$

For the left hand side we have

$$\begin{aligned}\liminf \int_A (f_i+g)(x)dm(x) &= \liminf\left(\int_A f_i(x)dm(x) + \int_A g(x)dm(x)\right) \\ &= \liminf \int_A f_i(x)dm(x) + \liminf \int_A g(x)dm(x) \\ &= \left(\liminf \int_A f_i(x)dm(x)\right) + \int_A g(x)dm(x).\end{aligned}$$

Since the integral of $g$ is on both sides of the inequality we can cancel it to get

$$\int_A f(x)dm(x) \leq \liminf \int_A f_i(x)dm(x).$$

Next we note that $|f_i| \leq g$ implies that $f_i \leq g$ and $g - f_i \geq 0$. Applying Fatou's Lemma to this sequence we get

$$\int_A \liminf(g - f_i)(x)dm(x) \leq \liminf \int_A (f_i + g)(x)dm(x).$$

Following similar reasoning as above we see that the left hand side of the inequality becomes

$$\int_A g(x)dm(x) - \int_A f(x)dm(x)$$

and the right hand side is

$$\int_A g(x)dm(x) + \left(\liminf \int_A -f_i(x)dm(x)\right).$$

Again the integrals of $g$ cancel out so we have

$$\begin{aligned}-\int_A f(x)dm(x) &\leq \liminf \int_A -f_i(x)dm(x) \\ &= -\limsup \int_A f_i(x)dm(x).\end{aligned}$$

This becomes

$$\int_A f(x)dm(x) \geq \limsup \int_A f_i(x)dm(x)$$

which completes the proof. $\boxed{13}$