Math 6210 - Homework 5

Try to finish before the Thanksgiving break.

Let $([0,1], \mathfrak{M}, m)$ be the standard Lebesgue measure space on $[0,1]$ and let $L^2[0,1] = L^2(m)$ (using the notation from Rudin).

1. Define $T : L^2[0,1] \to L^2[0,1]$ by $(Tf)(t) = tf(t)$. Show that:
   (a) T is bounded and symmetric with $\|T\| = 1$;
   (b) has no eigenvalues;
   (c) and $T - \lambda I$ is surjective if and only if $\lambda \notin [0,1]$.

Now let $([0,1]^n, \mathfrak{M}, m)$ be Lebesgue measure on the cube in $\mathbb{R}^n$ and let $L^2([0,1]^n) = L^2(m)$. We want to find an orthonormal basis for $L^2([0,1]^n)$. Our strategy will be the same as it was for $L^1[0,1]$.

We can also define Fourier series for periodic functions on $\mathbb{R}^n$. (If you like you can simplify things by assuming $n = 2$.)

We need a bit of notation. If $\boldsymbol{\xi} \in \mathbb{Z}^n$ and $\mathbf{x} \in \mathbb{R}^n$ then $\boldsymbol{\xi} \cdot \mathbf{x} = \xi_1 x_1 + \cdots + \xi_n x_n$. Let $|\boldsymbol{\xi}|_\infty = \max\{|\xi_1|, \ldots, |\xi_n|\}$.

A function $f : \mathbb{R}^n \to \mathbb{C}$ is periodic if $f(\mathbf{x}) = f(\mathbf{x} + \boldsymbol{\xi})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\xi} \in \mathbb{Z}^n$. Let $C(\mathbb{T}^n)$ be continuous periodic functions on $\mathbb{R}^n$. Of course, any function in $C(\mathbb{T}^n)$ can be restricted to a function in $L^2([0,1]^n)$. We want to show that the functions $e_{\boldsymbol{\xi}}(\mathbf{x}) = e^{2\pi\imath \boldsymbol{\xi} \cdot \mathbf{x}}$ are an orthonormal basis for $L^2([0,1]^n)$.

2. Show that $(e_{\boldsymbol{\xi}_0}, e_{\boldsymbol{\xi}_1}) = 1$ if $\boldsymbol{\xi}_0 = \boldsymbol{\xi}_1$ and $(e_{\boldsymbol{\xi}_0}, e_{\boldsymbol{\xi}_1}) = 0$ if $\boldsymbol{\xi}_0 \neq \boldsymbol{\xi}_1$.
3. Show that $e_{\boldsymbol{\xi}}(\mathbf{x}) e_{\boldsymbol{\xi}}(\mathbf{y}) = e_{\boldsymbol{\xi}}(\mathbf{x} + \mathbf{y})$.
4. (Optional) Let
$$D_N(x) = \sum_{k=-N}^{N} e^{2\pi\imath kx}$$
and show that
$$D_N(x) = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}.$$
5. (Optional) Let
$$K_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x)$$
and show that
$$K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2.$$

6. Let

$$\mathbf{K}_N(\mathbf{x}) = \frac{1}{N^n}\prod_{j=1}^{n}\sum_{M=0}^{N-1}\sum_{k=-M}^{M} e^{2\pi\imath k x_j}.$$

Show that

$$\mathbf{K}_N(\mathbf{x}) = \frac{1}{N^n}\prod_{j=1}^{n}\left(\frac{\sin(N\pi x_j)}{\sin(\pi x_j)}\right)^2$$

and that $\mathbf{K}_N(\mathbf{x})$ is a finite linear combination of $e_{\boldsymbol{\xi}}(\mathbf{x})$. Conclude that

$$f_N(\mathbf{x}) = \int_{[0,1]^n} f(\mathbf{y})\mathbf{K}_N(\mathbf{x}-\mathbf{y})dm(\mathbf{y})$$

is a finite linear combination of $e_{\boldsymbol{\xi}}(\mathbf{x})$.

7. Periodic functions $\Phi_k \in C(\mathbb{T}^n)$ form an *approximate identity* if

   (a) $\int_{[0,1]^n} \Phi_k(\mathbf{x})dm(\mathbf{x}) = 1$;

   (b) $\sup_n \int_{[0,1]^n} |\Phi_k(\mathbf{x})|dm(\mathbf{x}) < \infty$;

   (c) For all $\delta > 0$, $\int_{1/2>|\mathbf{x}|>\delta} |\Phi_k(\mathbf{x})|dm(\mathbf{x}) \to 0$.

   If Φ_k is an approximate identity and f is continuous show that

$$\int_{[0,1]^n} f(\mathbf{y})\Phi_k(\mathbf{x}-\mathbf{y})dm(\mathbf{y}) \to f(\mathbf{x})$$

uniformly.

8. Show that $\mathbf{K}_N(\mathbf{x})$ is an approximate identity and therefore if f is in $C(\mathbb{T}^n)$ show that f_N converges to f uniformly. Conclude that the $e_{\boldsymbol{\xi}}$ are an orthonormal basis for $L^2([0,1]^n)$.

   Hint: If $[a,b]$ is an interval show that

$$\int_{[a,b]\times[-1/2,1/2]^{n-1}} \mathbf{K}_N(\mathbf{x})dm(\mathbf{x}) = \int_a^b K_N(x)dm(x)$$

and therefore for all $\delta > 0$ we have

$$\lim_{N\to\infty}\int_{[\delta,1/2]\times[-1/2,1/2]^{n-1}} \mathbf{K}_N(\mathbf{x})dm(\mathbf{x}) \to 0$$

with the same statement holding if we replace $[\delta, 1/2]$ with $[-1/2, -\delta]$. Use the fact that the set of $\mathbf{x}$ with $1/2 > |\mathbf{x}| > \delta$ can be covered by $2n$ sets of the form $[\delta', 1/2] \times [-1/2, 1/2]^{n-1}$ or $[-1/2, -\delta'] \times [-1/2, 1/2]^{n-1}$ for some $\delta' > 0$ to show that $\mathbf{K}_N(\mathbf{x})$ satisfies (c).

9. Let H^n_s be sequences $u_{\boldsymbol{\xi}}$ indexed by $\boldsymbol{\xi} \in \mathbb{Z}^n$ such that

$$\sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} (1+|\boldsymbol{\xi}|^2)^s |u_{\boldsymbol{\xi}}|^2 < \infty.$$

Show that if $s \geq \lfloor n/2 \rfloor + 1$ and $u \in H^n_s$ then the series $\sum |u_{\boldsymbol{\xi}}|$ converges. Conclude that $\sum u_{\boldsymbol{\xi}} e_{\boldsymbol{\xi}}(\mathbf{x})$ converges to a continuous function. (Hint: Write $|u_{\boldsymbol{\xi}}| = (1+|\boldsymbol{\xi}|^2)^{-s/2}(1+|\boldsymbol{\xi}|^2)^{s/2}|u_{\boldsymbol{\xi}}|$ and apply the Cauchy-Schwarz inequality. You will use the assumption that $s \geq \lfloor n/2 \rfloor + 1$ to show that $\sum (1+|\boldsymbol{\xi}|^2)^{-s}$ converges.)

   If $s \geq \lfloor n/2 \rfloor + m + 1$ show $\sum u_{\boldsymbol{\xi}} e_{\boldsymbol{\xi}}(\mathbf{x})$ has partial derivatives of all order $\leq m$.

10. We can make H_s^n into a Hilbert space by defining the inner product

$$(u, v)_s = \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} (1 + |\boldsymbol{\xi}|^2)^s u_{\boldsymbol{\xi}} \overline{v}_{\boldsymbol{\xi}}.$$

If $s > t$ show that the inclusion of H_s^n in H_t^n is a compact operator.