Math 6210 - Homework 5
Due at 4 PM on 11/10/05

From Rudin: Chapter 5, # 1,2,3,4,5

All of the problems will examine linear operators on the Hilbert space $H = \ell^2(\mathbb{Z})$. Recall that $f \in H$ if f is a function

$$f : \mathbb{Z} \longrightarrow \mathbb{C}$$

with $\sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty$ and the inner product of $f, g \in H$ is

$$(f, g) = \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)}.$$

We also recall the definition of the adjoint T^* of a linear operator T. If T is bounded then we saw in class that for any $g \in H$ the linear map

$$f \mapsto (Tf, g)$$

is bounded. Therefore there is a unique element $g' \in h$ such that $(Tf, g) = (f, g')$ for all $f \in H$. We define $T^*g = g'$.

If T is unbounded it will in general only be defined on a subspace of H which we denote $\operatorname{dom} T$. Then given any $g \in H$ the map

$$f \mapsto (Tf, g)$$

is only defined for $f \in \operatorname{dom} T$. Furthermore it will not always be bounded. The subset of H where this map is bounded is the domain of the adjoint, $\operatorname{dom} T^*$. Then for $g \in \operatorname{dom} T^*$ there is a unique g' such that $(Tf, g) = (f, g')$ for all $f \in \operatorname{dom} T$. As in the bounded case we set $T^*g = g'$.

1. Define $T : H \longrightarrow H$ by $(Tf)(n) = f(n+1)$. Show that T is bounded and that $\|T\| = 1$. Find the adjoint T^* of T.

2. In this problem we will examine an unbounded operator S. Let $\operatorname{dom} S$ be the set of compactly supported functions in H. That is $f \in \operatorname{dom} S$ if $f(n) \neq 0$ for only finitely many $n \in \mathbb{Z}$. We then set $(Sf)(n) = nf(n)$.

   (a) Show that S is unbounded.

   (b) Show that

   $$\operatorname{dom} S^* = \left\{ g \in H \text{ such that } \sum_{n\in\mathbb{Z}} n^2 |g(n)|^2 < \infty \right\}.$$

   Here are some hints. If the sum is $< \infty$ apply Hölders inequality. If the sum is infinite we need to show that $g \notin \operatorname{dom} S^*$ by showing that

   $$f \mapsto (Sf, g)$$

is not bounded. To do this define $f_k \in \operatorname{dom} S$ with k a positive integer by

$$f_k(n) = \begin{cases} ng(n) & \text{if } |n| \le k \\ 0 & \text{if } |n| > k \end{cases}$$

and observe that $(Sf_k, g) = \|f_k\|_2^2$. Note that $\|f_k\| \to \infty$ and therefore

$$\frac{|(Sf_k, g)|}{\|f_k\|_2} \to \infty.$$

(c) Show that $(S^*g)(n) = ng(n)$ for $g \in \operatorname{dom} S^*$. In particular $S = S^*$ on $\operatorname{dom} S$ (which is contained in $\operatorname{dom} S^*$.) However S is *not* self-adjoint since S and S^* have different domains.