Math 6510 - Homework 6
Due at 4 PM on 10/22/04

1. Define $\mathbb{C}P^1$ to be equivalence classes of points in $\mathbb{C}^2\backslash\{0\}$ with the equivalence relation $(z, w) \sim (z', w')$ if there exists a $\lambda \in \mathbb{C}$ with $(z, w) = \lambda(z', w')$. Let

$$\pi : \mathbb{C}^2\backslash\{0\} \longrightarrow \mathbb{C}P^1$$

be the quotient map.

   (a) Show that $\mathbb{C}P^1$ has a differentiable structure such that π is a submersion. (Hint: There is a diffeomorphim from $\mathbb{C}$ to the set of points in $\mathbb{C}^2$ with $w = 1$. Show that the composition of this map with π is (the inverse of) a chart. Do the same thing with $z = 1$. These two charts cover $\mathbb{C}P^1$.)

   (b) Let $f : \mathbb{C}^2\backslash\{0\} \longrightarrow \mathbb{C}^2\backslash\{0\}$ be a smooth map with $f(\lambda z, \lambda w) = \mu f(z, w)$ where $\mu \in \mathbb{C}$ is a constant. Show that there is a unique smooth map $\bar{f} : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ with $\pi \circ f = \bar{f} \circ \pi$.

   (c) Given a polynomial
$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$
   define
$$P(z, w) = (z^n + a_{n-1}z^{n-1}w + \cdots a_0w^n, w^n).$$
   Show that $P(\lambda z, \lambda w) = \lambda^n P(z, w)$ and that therefore exists a smooth map $\bar{P}$ as in (b).

   (d) Show that $\overline{P}$ has mod 2 degree n by showing that
$$P_t(z, w) = (z^n + t(a_{n-1}z^{n-1}w + \cdots + a_0w^n), w^n)$$
   defines a homotopy of P to the map
$$(z, w) \mapsto (z^n, w^n)$$
   and that this descends to a homotopy of $\bar{P}$.

   (e) Conclude that if n is odd there is a point $[(z, w)] \in \mathbb{C}P^1$ such that $\bar{P}([(z, w)]) = [(0, 1)]$ and that therefore there is a $z \in \mathbb{C}$ such that $p(z) = 0$ for the original polynomial p.

2. 2.4 #4,5,6; 3.2 #2, 4 (For 2.4 #5: A manifold X is *contractable* if there is a homotopy $F : X \times I \longrightarrow X$ with $f_0 = \mathrm{id}$ and f_1 a constant map.)