**Final**

Due at 4 PM on 12/17/04

Do 5 of the 10 problems in 1 1/2 hours.

1. Let $M \subset \mathbb{R}^n$ be a compact manifold without boundary. Prove that there is $\epsilon > 0$ such that for any manifold $X$ the following holds: any two smooth maps

$$f, g : X \to M$$

with $|f(x) - g(x)| < \epsilon$ for every $x \in X$ are homotopic. Find an explicit $\epsilon > 0$ for the case when $M$ is the unit sphere $S^{n-1}$.

2. Compute the Lefschetz number of the mapping

$$f_k : \mathbb{C}P^n \to \mathbb{C}P^n$$

given by

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0^k : x_1^k : \cdots : x_n^k]$$

for $k = 1, 2, \cdots$.

3. Let $p$ be a homogeneous polynomial in $k$ real variables. This means that

$$p(tx_1, tx_2, \cdots, tx_k) = t^m p(x_1, x_2, \cdots, x_k)$$

for all $t, x_1, x_2, \cdots, x_k \in \mathbb{R}$. Prove that for $a \neq 0$ the set

$$\{x \in \mathbb{R}^k | p(x) = a\}$$

is a submanifold of $\mathbb{R}^k$.

4. Let $X$ and $Y$ be two submanifolds of $\mathbb{R}^n$. Show that for almost every $a \in \mathbb{R}^n$ the translate

$$X + a = \{x + a | x \in X\}$$

is transverse to $Y$.

5. Prove that the Euler characteristic of the orthogonal group $O(n)$ is 0 for $n \geq 1$.

6. Let $V$ be the vector field in the complex plane $\mathbb{C}$ given by

$$V(z) = z^m$$

for some $m = 1, 2, \cdots$, where we use the standard identification $T_z\mathbb{C} = \mathbb{C}$. Compute the index of $V$ at the origin.

7. Let $\omega$ be the 2-form on $R^3$ given by

$$\omega(x, y, z) = z dx \wedge dy + y dz \wedge dx + x dy \wedge dz$$

and define the map $f : \mathbb{R}^2 \longrightarrow R^3$ by

$$f(u, v) = (u, v, u^2 + v^2).$$

Calculate $f^*\omega$.

8. Let $\omega$ be the 1-form on $\mathbb{R}^2 - \{0\}$ given by

$$\omega(x, y) = \frac{-y dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2}$$

   (a) Calculate $\int_C \omega$ for any circle $C$ centered at the origin.

   (b) Prove that in the half-plane $x > 0$ there is a smooth function $f$ such that $\omega = df$.

   (c) Prove that there is no smooth function $f$ defined on $\mathbb{R}^2 - \{0\}$ such that $\omega = df$.

9. Give an example (with proof) of a nonintegrable plane field.

10. Compute de Rham cohomology of the 3-torus $S^1 \times S^1 \times S^1$.