MATH 7280 Operator Theory-Spring 2006
Instructor: Marian Bocea

Homework Assignment # 2
(Due March 27, 2006)

1. Consider the space $C([0,1])$ of real continuous functions defined on $[0,1]$ endowed with the norm $\|f\|_{C([0,1])} := \sup_{x\in[0,1]} |f(x)|$.

(i) Let $f \in C([0,1])$, $\{f_n\} \subset C([0,1])$ be such that $f_n \rightharpoonup f$ weakly. Show that $f_n(x) \to f(x)$ for any $x \in [0,1]$.

(ii) Let $\{f_n\} \subset C([0,1])$, where

$$f_n(x) = \begin{cases} n - n^2 \left|x - \frac{1}{n}\right| & \text{if } x \in \left[0, \frac{2}{n}\right], \\ 0 & \text{if } x \in \left(\frac{2}{n}, 1\right] \end{cases}$$

Show that $f_n(x) \to 0$ for any $x \in [0,1]$, but $\{f_n\}$ does not converge to 0 weakly.

2. (i) Let E be a Banach space, and $\{f_n\} \subset E'$ be such that for any $x \in E$ the sequence $\{\langle f_n, x\rangle_{E',E}\}$ converges. Prove that there exists $f \in E'$ such that $f_n \rightharpoonup f$ weakly $\star$ in E'.

(ii) Let E be a reflexive Banach space, and let $\{x_n\} \subset E$ be such that for any $f \in E'$ the sequence $\{\langle f, x_n\rangle_{E',E}\}$ converges. Prove that there exists $x \in E$ such that $x_n \rightharpoonup x$ weakly in E.

3. Let X, Y be two Banach spaces, and $T \in \mathcal{L}(X,Y)$ be a linear and continuous operator. Prove that

(i) if either X or Y is reflexive then T is *sequentially weakly compact*:

$\{x_n\}$ bounded in $X \Longrightarrow \{Tx_n\}$ contains a weakly convergent subsequence in Y.

(ii) if X is reflexive, then the set

$$T\left(\{x \in X \ : \ \|x\|_E \le 1\}\right)$$

is closed in Y (with respect to the strong topology).