Questions in Geometric Group Theory

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Notes:

I regret that I am unable to offer 1M for solutions.

Disclaimer: Names in parentheses reflect the person I heard the question from. The actual source may be different. Any corrections, new questions, additional references, clarifications, solutions and info welcome.

For questions with a decidedly more combinatorial flavor, see http://www.grouptheory.org

1 Hyperbolic Groups

1.1 Characterization and Structure

Q 1.1. *Suppose G admits a finite $K(G,1)$. If G does not contain any Baumslag-Solitar subgroups $BS(m,n)$, is G necessarily hyperbolic? If G embeds in a hyperbolic group, is it hyperbolic?*

The answer is no if the "finite $K(G,1)$" assumption is replaced by "finitely presented" by [Brady].

Remarks: $BS(m,n) =< x,y|x^{-1}y^m x = y^n >$ for $m,n \neq 0$. One considers $\mathbb{Z}\times\mathbb{Z}$ as a Baumslag-Solitar group $BS(1,1)$.

The answer is not known even when $Y = K(G,1)$ is a non-positively curved 2-complex. In that case, the question becomes: If the universal cover $\tilde{Y}$ contains a flat, does G contain $\mathbb{Z}\times\mathbb{Z}$?

It is possible that there are counterexamples that contain certain infinite quotients of Baumslag-Solitar groups. A modification of the above question would be:

Q 1.2. *Suppose G admits a finite $K(G,1)$, does not contain $\mathbb{Z}\times\mathbb{Z}$, and whenever $x \in G$ is an infinite order element such that x^m and x^n are conjugate, then $|m| = |n|$. Is G hyperbolic?*

All known constructions of hyperbolic groups of large (rational, or integral for torsion-free groups) cohomological dimension use arithmetic groups as building blocks. See [Charney-Davis, Strict Hyperbolization], [Gromov, Asymptotic Invariants, Ch.7]. Gromov loosely conjectures that this is always the case. The following is an attempt to make the question precise.

Q 1.3. *For every $K > 0$ there is $N > 0$ such that every word-hyperbolic group G of rational cohomological dimension $\geq N$ contains an arithmetic lattice of dimension $\geq K$.*

Note: An earlier formulation of this question required $N = K$ for K large. A counterexample to the stronger formulation was found by [Mosher-Sageev], see `http://andromeda.rutgers.edu/ ~mosher/HighDHyp.ps`. See also [Gromov, Asymptotic Invariants, 7.VI]. One might want to allow certain quotients of arithmetic groups as well.

The following is another instance of Gromov's loose conjecture.

Q 1.4. *Conjecture (Davis): All hyperbolic Coxeter groups have uniformly bounded rational (or virtual) cohomological dimension.*

[Moussong] has a criterion for recognizing word-hyperbolicity from the Coxeter diagram.

Update (May 2002): T. Januszkiewicz and J. Swiatkowski have constructed hyperbolic Coxeter groups of arbitrarily large dimension. This answers both questions above. The preprint is available at

`http://www.math.uni.wroc.pl/~tjan/Papers/submitted.html`

Q 1.5. *(Davis) If G is word-hyperbolic, does the Rips complex $P_d(G)$ have an equivariant negatively curved metric for d sufficiently large?*

A potential counterexample is the mapping torus of a hyperbolic automorphism of a free group.

1.2 Subgroups of Hyperbolic Groups

Q 1.6. *(Gromov) Does every 1-ended word-hyperbolic group contain a closed hyperbolic surface subgroup?*

Q 1.7. *(Gromov) For a given n is there an example of a hyperbolic group of dimension n in which every infinite index subgroup is free? Or in which there are no (quasi-convex) subgroups with codimension $\leq k$ for a given $k \leq n-2$.*

Q 1.8. *(Swarup) Suppose H is a finitely presented subgroup of a word-hyperbolic group G which has finite index in its normalizer. Assume that there is $n > 0$ such that the intersection of n distinct conjugates of H is always finite. Is H quasi-convex in G?*

The converse is a theorem of [Gitik-Mitra-Rips-Sageev]. A special case worth considering is when G splits over H when Gersten's converse of the combination theorem might be helpful.

Remark(Gitik): The problem is open even when H is malnormal in G.

Q 1.9. *(Mitra) Let X_G be a finite 2-complex with fundamental group G. Let X_H be a cover corresponding to the f.p. subgroup H. Let $I(x)$ denote the injectivity radius of X_H at x. Does $I(x) \to \infty$ as $x \to \infty$ imply that H is quasi-isometrically embedded in G ?*

A positive answer to the above question for G hyperbolic would imply a positive answer to Q 1.8.

Q 1.10. *(Canary) Let G be word-hyperbolic and H a f.p. subgroup of G. Suppose that for all $g \in G$ there is $n > 0$ such that $g^n \in H$. Does it follow that H has finite index in G?*

Yes if H is quasi-convex, since then $\Lambda(H) = \Lambda(G)$.

Q 1.11. *(Whyte) Let Γ be a 1-ended hyperbolic group which is not virtually a surface group. Can every infinite index subgroup be free?*

Q 1.12. *(Whyte) Let Γ be a 1-ended hyperbolic group. Can a finite index subgroup of Γ be isomorphic to a subgroup of Γ of infinite index?*

1.3 Relative Questions

Q 1.13. *(Swarup) Prove the combination theorem for relatively hyperbolic groups.*

Q 1.14. *(Swarup) There is a theorem of Bowditch saying that if G acts on a compact metrizable space X so that the action is properly discontinuous and cocompact on triples, then G is a hyperbolic group and X is its boundary. State and prove the analog in which one allows parabolics.*

1.4 Residual Finiteness

Q 1.15. *Is every word-hyperbolic group residually finite?*

Notes:

a) D. Wise has constructed a finite 2-dimensional locally CAT(0) complex whose fundamental group is not RF.

b) Z. Sela showed that torsion-free word-hyperbolic groups are Hopfian.

c) M. Kapovich pointed out that there are non-linear word-hyperbolic groups. Start with a lattice G in quaternionic hyperbolic space and adjoin a "random" relation to get the desired group. Super-rigidity [Corlette, Gromov-Schoen] says that every linear representation of G is either faithful or has finite image.

Q 1.16. *(Dani Wise)* Let G^n denote the Cartesian product of n copies of the group G, and let $rank(G^n)$ be the smallest number of generators of G^n.
Conjecture. *If G is word-hyperbolic then* $\lim_{n\to\infty} rank(G^n) = \infty$.

The Conjecture is true if G is finite and nontrivial, as can be seen by a simple pigeon-hole argument. It is also true if G has a quotient for which the conjecture holds. In particular, it is true for groups that have a proper finite index subgroup.

The Conjecture is false if there is an epimorphism $G \to G \times G$. There is an example (Wise) of a 2-generator infinitely presented $C'(\frac{1}{6})$ small cancellation group where the conjecture fails.

Q 1.17. *(Dani Wise) Find "nice" (e.g. $CAT(0)$, automatic,...) groups where this conjecture fails.*

1.5 Algorithms

Q 1.18. *(Epstein) Let G be a word-hyperbolic group and ∂G its boundary. Is there an algorithm to compute $\check{H}^i(\partial G) \cong H^{i+1}(G, \mathbb{Z}G)$? In particular, is there an algorithm to decide whether $\check{H}^i(\partial G) \cong \check{H}^i(S^2)$ for all i?*

If ∂G has the cohomology of S^2 then it is homeomorphic to S^2 (Bestvina-Mess) and modulo a finite normal subgroup G is conjecturally commensurable to a hyperbolic 3-manifold group.

Remark (Epstein, Sela) There is an algorithmic procedure to determine the number of ends (i.e. $\check{H}^0(\partial G)$) of a hyperbolic group. First, one finds algorithmically an explicit δ (of δ-hyperbolicity). Then one finds an automatic structure, from which it can be immediately read if the group is finite or 2-ended. One can run an (enumeration) machine that terminates if the group nontrivially splits over a finite subgroup, i.e. if it has infinitely many ends. Finally, there is a machine (Gerasimov) that terminates if the group is 1-ended. This is based on the property of 1-ended groups (proved by Bowditch) that there is a universal bound on the length of a shortest path connecting points on a sphere $S(R)$ at distance $< 100 + 100\delta$ and disjoint from the sphere $S(R - 10 - 10\delta)$.

Remark (Sela) For 1-ended torsion-free hyperbolic groups there is an algorithm based on Sela's work on the isomorphism problem to decide if the group splits over $\mathbb{Z}$. This algorithm can be "relativized" to find the JSJ decomposition as well. The details of this have not appeared. The case of groups with torsion is open.

1.6 Maps Between Boundaries

Q 1.19. *(M. Mitra) Let G be a word-hyperbolic group and H a word-hyperbolic subgroup. Does the inclusion $H \to G$ extend to a continuous map between the boundaries $\partial H \to \partial G$?*

A theorem of Cannon-Thurston says that this is the case when G is the fundamental group of a hyperbolic 3-manifold that fibers over S^1 and H is the group of the fiber. The map $S^1 \to S^2$ is surjective and it can be explicitely described in terms of the stable and unstable laminations of the monodromy. When the manifold is closed, the map is finite-to-1.

By a theorem of Mitra, if G is a graph of groups with one vertex group H, all vertex groups hyperbolic, and all edge-to-vertex monomorphisms quasi-isometric embeddings, then the answer is yes.

Q 1.20. *(Swarup) Suppose G is a hyperbolic group which is a graph of hyperbolic groups such that all edge to vertex inclusions are quasi-isometric embeddings. [Mitra] shows that for each vertex group V inclusion $V \hookrightarrow G$ induces a continuous Cannon-Thurston map $\partial V \to \partial G$. Describe the point-preimages. In particular, show that the map is finite-to-one.*

1.7 Miscellaneous

Q 1.21. *(Thurston) Is every closed hyperbolic 3-manifold finitely covered by one that fibers over the circle?*

Q 1.22. *(Jim Anderson) Can the fundamental group of a hyperbolic 3-manifold that fibers over the circle contain a subgroup that is locally free but not free?*

There are such hyperbolic 3-manifolds, so the negative answer to this question would provide a counterexample to Q 1.21.

Richard Kent constructs an infinite family of examples. See
`http://www.math.utexas.edu/~rkent/cuff`

Q 1.23. *(Ian Leary) Is there a version of the Kan-Thurston theorem using only CAT(-1) groups, or word hyperbolic groups? (The statement should be: for any finite simplicial complex X, there is a locally CAT(-1) polyhedral complex Y and a map $Y \to X$ that is surjective on fundamental groups and induces an isomorphism on homology for any local coefficients on X.)*

Q 1.24. *(Ian Leary) Is there any restriction on the homotopy type of the quotient $R_d(G)/G$ for G word hyperbolic?*

Here R_d is the Rips complex, and d should be taken to be large (so that $R_d(G)$ is a model for the universal proper G-space). Of course this space is a finite complex, but is there any other restriction on its homotopy type?

Background: Leary-Nucinkis proved that any complex has the homotopy type of $\underline{E}G/G$ for some discrete group G, where $\underline{E}G$ denotes the universal proper G-space. (This is a version of the Kan-Thurston theorem for $\underline{E}G$ instead of EG.)

Leary can answer the analogues of both questions above with CAT(0) in place of CAT(-1) or word hyp., as well as the 2-dimensional cases of both using either CAT(-1) groups or small cancellation groups. The trouble with getting to higher dimensions is that most proofs of Kan-Thurston type results use direct products to make higher dimensional groups.

2 $CAT(0)$ groups

Q 2.1. *(Swarup) Is there a proof of Johannson's theorem that $Out(\pi_1 M)$ is virtually generated by Dehn twists for M a Haken 3-manifold along the lines of Rips-Sela's theorem that $Out(G)$ is virtually generated by Dehn twists for hyperbolic G? Is this true for CAT(0) groups? In particular, if G is a CAT(0) group and $Out(G)$ is infinite, does G admit a Dehn twist of infinite order?*

By a Dehn twist we mean an automorphism ϕ of the following form: either G splits as $A *_C B$, $t \in C$ is central, $\phi(a) = a$ for $a \in A$ and $\phi(b) = t^{-1}bt$ for $b \in B$; or G is an HNN extension and ϕ is defined similarly.

Q 2.2. *(Gromov) If G admits a finite dimensional $K(G,1)$, does G act properly discontinuously by isometries on a complete CAT(0) space?*

Q 2.3. *(Eilenberg-Ganea) Is there a group G of cohomological dimension 2 and geometric dimension 3?*

Q 2.4. *(Whitehead) Is every subcomplex of an aspherical 2-complex aspherical?*

Q 2.5. *(Exercise in [Ballmann-Gromov-Schroeder, p.2]) Take a closed surface S of genus ≥ 2. Let $V = S \times S$ and let $\Sigma \subset V$ denote the diagonal. Let $\tilde{V}$ be a nontrivially ramified finite cover of V along Σ. Then $\tilde{V}$ has a natural piecewise hyperbolic CAT(0) metric. Show that $\tilde{V}$ admits no C^2-smooth Riemannian metric with curvature $K \leq 0$.*

Q 2.6. *Suppose a group G acts properly discontinuously and cocompactly by isometries on two CAT(0) spaces X and Y. Croke-Kleiner have examples where the boundaries ∂X and ∂Y are not equivariantly homeomorphic. Is there a compact metric space Z and cell-like maps $Z \to \partial X$, $Z \to \partial Y$?*

A surjective map between metric compacta is *cell-like* if each point preimage is cell-like. A compact metric space is cell-like if when embedded in the Hilbert cube I^∞ (or I^n if it is finite-dimensional) it is contractible in each of its open neighborhoods.

Q 2.7. *(D. Wise) Let G act properly discontinuously and cocompactly on a CAT(0) space (or let G be automatic). Consider two elements a, b of G. Does there exist $n > 0$ such that either the subgroup $\langle a^n, b^n \rangle$ is free or $\langle a^n, b^n \rangle$ is abelian?*

Q 2.8. *Do CAT(0) (or (bi)automatic) groups satisfy the Tits alternative?*

Q 2.9. *Does every Artin group have a finite $K(G,1)$?*

Yes for Artin groups of finite type (meaning that the associated Coxeter group is finite) by the work of Deligne.

Q 2.10. *(Bestvina) Let $L_1, L_2, \cdots, L_n$ be a finite collection of straight lines in the plane $\mathbb{R}^2$. Does the fundamental group of*

$$X = \mathbb{R}^2 \times \mathbb{R}^2 \setminus \bigcup_{i=1}^{n} L_i$$

admit a finite $K(\pi, 1)$?

By [Deligne] the answer is yes for simplicial arrangements. In general X is not aspherical, but in all examples one understands, X becomes aspherical after attaching finitely many ≥ 3-cells. It is not even known whether $\pi_1(X)$ is torsion-free. A similar question can be asked about finite hyperplane arrangements in $\mathbb{R}^m$.

Q 2.11. *(Eric Swenson) Let X be a $CAT(0)$ metric space and G a finitely generated group that acts properly discontinuously by isometries on X.*

1. Can G be an infinite torsion group? The expected answer is yes.

2. If the action is cocompact, can G contain an infinite torsion subgroup? The expected answer is no.

Q 2.12. *(Kim Ruane) Let G be a Coxeter group, e.g. right-angled, and assume that G acts properly discontinuously and by isometries on a $CAT(0)$ space X. How is X different from the Coxeter complex? Specifically:*

Suppose that H is a special subgroup of G. Is there a closed convex subset of X on which H acts cocompactly?

Q 2.13. *(Ruth Charney) Classify Coxeter groups up to isomorphism.*

Interesting examples of isomorphic Coxeter groups (and Artin groups) with non-isomorphic diagrams were given by Noel Brady, Jon McCammond, Bernhard Muehlherr, and Walter Neumann. In the opposite direction, conditions under which isomorphism of groups implies isomorphism of diagrams were given by Charney-Davis, David Radcliffe, Anton Kaul.

Q 2.14. *(Ruth Charney) Classify Artin groups up to isomorphism.*

Q 2.15. *(Ruth Charney) Are all finite type Artin groups linear?*

Braid groups are linear by the recent work of Bigelow and Krammer.

Update: A.M. Cohen and D.B. Wales and independently F. Digne show that the answer is yes. See `http://front.math.ucdavis.edu/math.GR/0010204` and `http://www.mathinfo.u-picardie.fr/digne`

Q 2.16. *(Ruth Charney) Are all [finite type] Artin groups $CAT(0)$?*

The answer is yes for small numbers of generators by the work of Krammer, Tom Brady, Jon McCammond. The question is open even for braid groups.

Q 2.17. *(Ruth Charney) Are all Artin groups automatic?*

Q 2.18. *(Ross Geoghegan) Let M be a proper $CAT(0)$ space. We say that M is almost geodesically complete if there is $R \geq 0$ such that for all $a, b \in M$ there is an infinite geodesic ray starting at a and passing within R of b.*

If $Isom(M)$ acts cocompactly on M, is M almost geodesically complete?

Note: It is a theorem of Ontaneda that if a discrete group acts cocompactly by isometries on M and if $H_c^*(M) \neq 0$ then M is almost geodesically complete.

Q 2.19. *(Dani Wise)* A *triplane* is a CAT(0) space obtained by gluing together three Euclidean half-planes along their boundaries. Say that a CAT(0) space X has *isolated flats* if X does not contain an isometrically embedded triplane.

Let G act on a CAT(0) space X. Recall that a subgroup H is quasiconvex relative to this action provided that for a point $x \in X$ there is a constant K such that for any two points in the orbit Hx, the geodesic connecting these points lies in a K-neighborhood of Hx.

Conjecture: *Let G act properly discontinuously and cocompactly on a CAT(0) space X with isolated flats. Let H be a finitely generated subgroup of G. Then the inclusion of H in G is a quasi-isometric embedding if and only if H is a quasiconvex subgroup relative to the action of G on X.*

Notes: Wise's, C.Hruska, has now proven this conjecture in the case that X is a CAT(0) 2-complex. A counterexample, without the isolated flats condition, is provided by $G = F_2 \times \mathbb{Z} =< a, b > \times < t >$ and $H =< at, bt >$ with the usual action of G on $(tree) \times \mathbb{R}$. H is not quasi-convex since the intersection $H \cap < a, b >$ is not finitely generated.

3 Free Groups

Q 3.1. *(Hanna Neumann Conjecture) If A and B are nontrivial subgroups of a free group, then $rk(A \cap B) - 1 \leq (rk(A) - 1)(rk(B) - 1)$.*

Hanna Neumann showed [Publ. Math. Debrecen 4 (1956), 186–189] $rk(A \cap B) - 1 \leq 2(rk(A) - 1)(rk(B) - 1)$, and R.G. Burns [Math. Z. 119 (1971), 121–130] strengthened the inequality to $rk(A \cap B) - 1 \leq 2(rk(A) - 1)(rk(B) - 1) - \min(rk(A) - 1, rk(B) - 1)$. In many cases, the conjecture holds, see Neumann, Walter D.: On intersections of finitely generated subgroups of free groups. Groups—Canberra 1989, 161–170, Lecture Notes in Math., 1456, Springer, Berlin, 1990. See also Dicks, Warren: Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture. Invent. Math. 117 (1994), no. 3, 373–389.

Q 3.2. *(Swarup) If G is a Fuchsian group, define $area(G)$ to be the area of the convex core of $\mathbb{H}^2/A$. Since $area(A) = 2\pi(rk(A)-1)$ for Fuchsian groups which are free, the Hanna Neumann Conjecture can be phrased in terms of free Fuchsian groups: $2\pi area(A \cap B) \leq area(A)area(B)$ whenever A and B are nontrivial free subgroups of a Fuchsian group. Prove such inequalities (possibly with a worse constant) for (not necessarily free) torsion-free quasi-convex subgroups of a quasi-convex Kleinian group in $\mathbb{H}^n$, where area is replaced by the n-dimensional volume of the convex core.*

Q 3.3. *(J. Cornick) a) Let G be a f.g. group such that every f.p. subgroup is free. Is G free?*

b) If G is f.g. and the homological dimension hd $G = 1$, is G free?

Note: Yes to a) implies yes to b). There are counterexamples to a) in which every f.p. subgroup is cyclic and torsion-free.

3.1 Limit groups (Zlil Sela)

The definition of limit groups involves group actions on $\mathbb{R}$-trees and is given by Sela in his work on equations and inequalities over free groups. It is a nontrivial theorem that a group G is a limit group iff it is ω-residually free, i.e. for any finite subset F of G there is a homomorphism from G to a free group whose restriction to F is injective. Every limit group is finitely presented.

Q 3.4. Conjecture. *A finitely generated group admits a free action on an $\mathbb{R}^n$-tree for some n iff it is a limit group.*

The "if" part follows from Sela's work.

Q 3.5. Conjecture. *Limit groups are $CAT(0)$.*

Q 3.6. *Characterize groups of the form $F_m *_{\mathbb{Z}} F_n$ which are limit groups.*

Q 3.7. *Characterize 1-relator groups which are limit groups.*

Q 3.8. *(Swarup) Show that a limit group is relative hyperbolic w.r.t its maximal abelian subgroups of rank bigger than one.*

Sela showed that limit groups that contain no $\mathbb{Z} \times \mathbb{Z}$ are hyperbolic.

3.2 Definable sets in F_n (Zlil Sela)

A subset S of F_n^k is *definable* if there is a first order sentence with k variables $x_1, x_2, \cdots, x_k$ which is true iff $x_1, x_2, \cdots, x_k \in S$.

Q 3.9. Conjecture. *Let $n > 1$. The set*

$$\{(x_1, x_2, \cdots, x_n) \in F_n^n | x_1, x_2, \cdots, x_n \textit{ is a basis of } F_n\}$$

is not definable.

Q 3.10. Conjecture. *The only definable subgroups of F_n are cyclic and the entire group.*

3.3 Genus in free groups (Zlil Sela)

The *genus* of an element x in the commutator subgroup of F_n is the smallest g such that we may write $x = [y_1, y_2][y_3, y_4] \cdots [y_{2g-1}, y_{2g}]$ for some $y_i \in F_n$. Thus there is a map of the surface of genus g and one boundary component into the rose with boundary corresponding to x. Applying an automorphism of the surface produces other ways of writing x as a product of g commutators, and we call all such *Nielsen equivalent*. One knows that there is a uniform bound $f(g)$ (independent of x) to the number of Nielsen equivalence classes in which an element of genus g can be written as a product of g commutators.

Q 3.11. *Give an explicit upper bound for $f(g)$.*

4 Transformation Groups

Q 4.1. *(de la Harpe) Is $PSL_2(\mathbb{R})$ a maximal closed subgroup of $Homeo_+(S^1)$?*

Q 4.2. *Is there a proper closed subgroup of $Homeo_+(S^1)$ that acts transitively on (unordered) 4-tuples? Or k-tuples ($k \neq 3$)? Relation to earthquakes?*

Q 4.3. *(Christian Skau) There are many examples of groups acting on non-homogeneous compacta with all orbits dense (boundaries of word-hyperbolic groups, limit sets of Kleinian groups). Are there such examples with group $= \mathbb{Z}$ (or amenable group)?*

Q 4.4. *(Swarup) Suppose G is a 1-ended finitely presented group that acts on a compact connected metric space X as a convergence group. What can be said about G if X has cut points? Does X have to be locally connected? The model theorem of [Bowditch] and [Swarup] says that if G is word hyperbolic, then X is locally connected and doesn't have cut points. In another interesting case, when G is a geometrically finite Kleinian group and X its limit set, cut points can arise, for example if G splits over a parabolic subgroup.*

5 Kleinian Groups

Q 5.1. *(Jim Anderson) If G is a group of isometries of $\mathbb{H}^n$, denote by $Ax(G)$ the set of axes of the elements of G. If G_1 and G_2 are finitely generated and discrete, does $Ax(G_1) = Ax(G_2)$ imply that G_1 and G_2 are commensurable?*

Yes if $n = 2$ and G_1, G_2 are arithmetic by [Long-Reid].

Q 5.2. *(G. Courtois, Gromov) Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold with $n \geq 3$. Does there exist a faithful non-discrete representation $\rho : \Gamma \to Isom_+(\mathbb{H}^n)$ (same n)?*

The answer is yes (for certain manifolds) if the target is $Isom_+(\mathbb{H}^{n+1})$ (Kapovich – use bending deformations), and also by Goldman's thesis if $n = 2$. In the case of n=3 (Kapovich, Reid), if $M = \mathbf{H}^3/\Gamma$ and the trace-field of Γ admits a real galois conjugate, then this can be used to induce a non-discrete faithful representation of Γ with real character. For example any arithmetic Kleinian group derived from a quaternion algebra over a field of degree at least 3 has such a real representation.

Q 5.3. *(Ed Taylor) Does there exist a constant $c > 0$ such that the limit set of every non-classical Schottky group has Hausdorff dimension $\geq c$.*

Q 5.4. *(Marden Conjecture) A torsion-free f.g. Kleinian group in dimension 3 is topologically tame, i.e. the quotient 3-manifold is the interior of a compact 3-manifold.*

Q 5.5. *(Misha Kapovich) Suppose that G is a finitely generated Kleinian group in $Isom(\mathbb{H}^n)$. Is it true that*

$$\delta(G) \leq vcd(G)$$

with equality iff G preserves a totally geodesic subspace $\mathbb{H}^k \subset \mathbb{H}^n$ so that $\mathbb{H}^k/G$ is compact?

Here $\delta(G)$ is the exponent of convergence of G and vcd is the virtual cohomological dimension. Note that the answer is positive for geometrically finite groups.

A weaker form of this question is:

Q 5.6. *(Misha Kapovich) Is there $\epsilon > 0$ so that $\delta(G) < \epsilon$ implies that G is virtually free?*

Q 5.7. *(Misha Kapovich) Is there a finitely-generated discrete subgroup of $SO(n, 1)$ whose action on the limit set is not ergodic? Is not recurrent?*

Note that there are examples of finitely generated discrete subgroups of $SU(2, 1)$ which do not act ergodically on the limit set (however in that example the action is recurrent).

Q 5.8. *(Misha Kapovich – a variation on a question of Bridson). Suppose that $G \subset SL(n, \mathbb{R})$ is a discrete subgroup of type FP_∞ (over $\mathbb{Z}$). Is it true that G contains only finitely many G-conjugacy classes of finite subgroups?*

Note that there are examples (Feighn-Mess) where G is a linear group of type FP_k for arbitrarily large (but finite) k so that G contains infinitely many conjugacy classes of finite order elements.

Update: This problem is completely solved (the answer is no) by Ian Leary and Brita Nucinkis,

`ftp://byrd.math.uga.edu/pub/html/archive/leary-nucinkis.html`

6 Finiteness properties

Q 6.1. *(Ian Leary) Suppose G is virtually of type FP over the field F_p of p elements, and let g be an element of order p. Is the centralizer of g in G also virtually of type FP over F_p?*

If G acts cocompactly on an F_p-acyclic space X, then the centralizer of g acts cocompactly on X^g = the set of fixed points of g, which is also F_p-acyclic by Smith theory. So in this case the answer is yes.

Q 6.2. *(Ian Leary) Is there a group of finite vcd that does not act with finite stablizers on an acyclic complex of dimension equal to its vcd?*

This question is sometimes called Brown's conjecture, although more often Brown's conjecture is taken to be the question of whether there is a universal proper G-space of dimension equal to the vcd. The answer to this latter question is "no" — Leary and Nucinkis have examples for each n of groups for which the vcd is $2n$ but the minimal dimension of the universal proper G-space is $3n$.

Q 6.3. *(Peter Kropholler) If G is FP over the rationals, is there a bound on the orders of finite subgroups of G?*

Kropholler showed that this was the case for any G that is both (a) of finite rational cohomological dimension and (b) type FP_∞ over the integers.

(Leary) This can be extended to the case when G is only assumed to be FP_n over the integers, for n = the rational coh. dim. of G.

7 Splittings, Accessibility, JSJ Decompositions

Q 7.1. *(Potyagailo)* Let $G \subset SO_+(n,1)$ be a geometrically finite, torsion-free, non-elementary Kleinian group. Say that G splits over an essentially non-maximal parabolic subgroup $C \subset G$ as $G = A *_C B$ (resp. $G = A*_C$) if C is contained in a parabolic subgroup C' of higher rank such that C' is not conjugate to a subgroup of A or B.

Is the following true: G is co-Hopfian iff it does not split over trivial or essentially non-maximal parabolic subgroup? Note that the latter condition is equivalent to saying that the parabolic splittings are K-acylindrical (in the sense of Z.Sela) for a uniformly bounded K.

Q 7.2. *(Swarup)* Let G be a finitely presented group. Consider a maximal graph of groups decomposition of G with finite edge groups and pass to the collection of vertex groups. For each vertex group consider a maximal graph of groups decomposition with 2-ended edge groups and pass to the collection of vertex groups. Then split again along finite groups, then along two-ended groups etc.

Conjecture 1: *There is a finitely presented group for which this process never terminates.*

Conjecture 2 (Strong Accessibility): *For hyperbolic groups (and for CAT(0) groups) this process always terminates.*

Delzant-Potyagailo proved Strong Accessibility for hyperbolic groups without 2-torsion.

Note: For CAT(0) groups it would be natural to allow splittings over virtually abelian subgroup in the process. For general f.p. groups splittings over slender (small?) subgroups should be allowed.

Q 7.3. *(Sageev) Is there a f.p. 1-ended group G with $G \cong G*_{\mathbb{Z}}$?*

Note (Mitra) that such G could not be co-Hopfian. In particular, G could not be torsion-free hyperbolic, by a theorem of Z. Sela.

Q 7.4. *(Swarup) Show that the JSJ decomposition of a Haken 3-manifold M depends only on $\pi_1 M$ and not on the peripheral structure. Deduce Johannson's deformation theorem: A homotopy equivalence $M \to N$ between Haken manifolds can be deformed so it induces a homeomorphism between non-SFS components of the JSJ decompositions and induces a homotopy equivalence between SFS pieces.*

Update: This is answered by Scott-Swarup.

Q 7.5. *(Papasoglou) Is there a f.p. torsion-free group G that does not split over a virtually abelian subgroup, but has infinitely many splittings over F_2?*

This question is motivated by the fact that an irreducible atoroidal closed 3-manifold has only finitely many incompressible surfaces of any fixed genus.

8 General Questions

Q 8.1. *(Eilenberg-Ganea) Is there a group G of cohomological dimension 2 and geometric dimension 3?*

Q 8.2. *(Shalen) If a finitely presented group G acts nontrivially (i.e. without global fixed points) on an $\mathbb{R}$-tree, does it act nontrivially on a simplicial tree?*

Q 8.3. *(Mohan Ramachandran)* See [Napier-Ramachandran] for motivation. Consider the following two properties of a finitely presented group G:

(A) G virtually splits, i.e. some finite index subgroup of G admits a nontrivial action on a simplicial tree.

(B) Let X be a finite complex with fundamantal group G. Then some covering space of X has at least two ends.

Most garden-variety groups satisfy both (A) and (B). Groups that satisfy property (T) satisfy neither (A) nor (B).

To what extent are (A) and (B) equivalent?

The question makes sense for finitely generated groups as well.

Q 8.4. *(Bowditch) Let Γ be the Cayley graph of an infinite finitely generated group. Does there exist $K > 0$ such that for all $R > 0$ and all vertices $v \in \Gamma \setminus B(R)$ there is an infinite ray from v to ∞ which does not enter $B(R - K)$.*

Anna Erschler observed that this fails for the lamplighter group.

Q 8.5. *(Noel Brady) Are there groups of type F_n but not F_{n+1} ($n \geq 3$) which do not contain $\mathbb{Z} \times \mathbb{Z}$? All known examples contain $\mathbb{Z}^{n-1}$.*

Q 8.6. *(Olympia Talelli) Is there a torsion-free group G of infinite cohomological dimension such that there is n_0 with the property that if H is a subgroup of G with finite cohomological dimension cdH, then $cdH \leq n_0$.*

Q 8.7. *Can $\mathbb{Z}_{p^\infty}$ be embedded in an FP_∞-group? Or in an FP_3-group? Can an F_n-group be embedded in an F_{n+1}-group ($n \geq 2$)?*

Higman's celebrated embedding theorem states that a group embeds into a finitely presented group iff it has a recursive presentation. Any countable group embeds into a f.g. group.

Q 8.8. *Compute the asymptotic dimension of CAT(0) groups, $Out(F_n)$, mapping class groups, nonuniform lattices, Thompson's group. Is there a group of finite type whose asymptotic dimension is infinite?*

Note (Misha Kapovich): Gromov states that the answer to the last question is positive (and the group it does not admit a uniformly proper map into the Hilbert space).

8.1 Finite gap question (Jens Harlander)

Let G be a finitely generated group written as $G = F/N$ with F a free group of finite rank. Let Γ be a finite graph with $\pi_1(\Gamma) = F$ and let $\tilde{\Gamma}$ be the covering space of Γ with $\pi_1(\tilde{\Gamma}) = N$. Thus the deck group of $\tilde{\Gamma}$ is G. By $d_F(N)$ denote the smallest number of G-orbits of 2-cells one needs to attach to $\tilde{\Gamma}$ to make the space simply connected, and by $d_G(\frac{N}{[N,N]})$ denote the smallest number of G-orbits of 2-cells one needs to attach to $\tilde{\Gamma}$ to kill the first homology. Thus $d_F(N) < \infty$ iff G is finitely presented and $d_G(\frac{N}{[N,N]}) < \infty$ iff G is of type FP_2. There are examples (Bestvina-N. Brady) of groups of type FP_2 which are not finitely presented. A well-known question, stemming from the 1965 work off C.T.C. Wall, becomes:

Q 8.9. *Is there an example of a finitely presented group $G = F/N$ such that $d_G(\frac{N}{[N,N]}) < d_F(N)(< \infty)$?*

One approach to construct such an example is the following. Suppose $H = F/N$ is finitely presented and contains as subgroups groups of the form $C^n = C \times C \times \cdots \times C$ for all n. Define $G_n = H *_{C^n} H$ which is written as $F * F/N_n$ in the obvious way. If C is finite and perfect, then $d_{F*F}(\frac{N_n}{[N_n,N_n]})$ is independent of n.

Q 8.10. *If C is nontrivial and finite, is $\lim_{n\to\infty} d_{F*F}(N_n) = \infty$?*

9 2-complexes

Q 9.1. *(Whitehead) Is every subcomplex of an aspherical 2-complex aspherical?*

Q 9.2. *(Andrews-Curtis) If K and L are simple homotopy equivalent finite 2-complexes, can one transform K to L by a sequence of elementary collapses and expansions of 1- and 2-cells, and by sliding 2-cells (i.e. reattaching them by maps homotopic to the old attaching maps)?*

This is unknown even when K is contractible and L is a point.

Q 9.3. *(Wise) Is there a finite aspherical 2-complex X with $\pi_1(X)$ coherent and with $\chi(X) \geq 2$?*

10 Symmetric Spaces

Q 10.1. *(Igor Belegredek) Let X be a non-positively curved symmetric space. Find conditions on a group Γ so that the space of conjugacy classes of faithful discrete representations of Γ into the isometry group of X is compact (non-compact).*

Notes (Belegradek): Suppose a sequence of faithful discrete representations goes to infinity. That defines a natural isometric action of Γ on the asymptotic cone of X, that is an Euclidean Tits building according to Kleiner-Leeb. The action is small (in a certain sense) and has no global fixed point. Thus it is enough to understand what groups cannot (can) have such action. First example to look at is when X is a product of two rank one spaces, so the asymptotic cone of X is a product of two trees.

Q 10.2. *(Belegredek) Is there a 3-complex X (not necessarily aspherical) which is not homotopy equivalent to a 2-complex but $H^3(X; \{G\}) = 0$ for all local coefficients?*

The condition is equivalent to saying that X is dominated by a finite 2-complex, or that $id : X \to X$ is homotopic to a map into the 2-skeleton.

11 Quasi-Isometries

Q 11.1. *Study the quasi-isometry group $QI(\mathbb{R}^n)$. How big is it?*

Q 11.2. *(Kleiner) What are the quasi-isometries of the 3-dimensional group Sol?*

Q 11.3. *(Kleiner) What are the q.i's of the Gromov-Thurston examples of negatively pinched manifolds?*

Q 11.4. *(Feighn)* Let $\phi : F_n \to F_n$ be an automorphism of the free group F_n and let M_ϕ be its mapping torus.

Classify these groups up to quasi-isometry.

Note: It was shown by [Macura] that if two automorphisms have polynomial growths with distinct degrees, then their mapping tori are not quasi-isometric.

Q 11.5. *(Bridson) Same as above for automorphisms $\mathbb{Z}^n \to \mathbb{Z}^n$.*

Q 11.6. *(Bridson) Is $G \times \mathbb{Z}$ quasi-isometric to G for $G =$Thompson's group? Any f.g. group?*

12 Mapping Class Groups, $Out(F_n)$

Q 12.1. *(Levitt) Can every measured geodesic lamination with 2-sided leaves on a non-orientable compact hyperbolic surface be approximated by a simplicial measured geodesic lamination with 2-sided leaves?*

Remark: The subspace Ω of the sphere of geodesic measured laminations that contain 1-sided leaves is open and dense (of full measure, [Danthony-Nogueira]). More generally, understand the action of the mapping class group on the space of projectivized measured laminations. Is the action minimal on the complement of Ω.

Q 12.2. *(Lubotzky) Does $Out(F_n)$ have the congruence subgroup property?*

If $G \subset F_n$ is a characteristic subgroup of finite index, then kernels of homomorphisms to $Out(F_n/G)$ are called congruence subgroups.

Q 12.3. *(Kapovich) a) Let Σ_g be the closed orientable surface of genus g. Is there a faithful representation $\pi_1(\Sigma_g) \to MCG(\Sigma_h)$ into a mapping class group such that the image consists of pseudo-Anosov classes plus identity (for some $g, h > 1$)?*

b) Is there a 4-manifold M which is a surface bundle over a surface s.t. $\pi_1(M)$ is word-hyperbolic? Or so that M is hyperbolic?

Q 12.4. *(Bowditch) Is the Weil-Petersson metric on Teichmüller space hyperbolic? Is it quasi-isometric to the curve complex?*

The latter is word-hyperbolic by [Mazur-Minsky].

Update: J. Brock proved that W-P space is quasi-isometric to the "pants complex" (`http://www.math.uchicago.edu/~brock/home/text/papers/wp/www/wp.ps.gz`).

Brock-Farb have announced (October 2000) that the space is hyperbolic for a torus with ≤ 2 punctures and a sphere with ≤ 5 punctures, but not hyperbolic for more complicated surfaces. A related result was announced by Sumio Yamada: the completion of the Weil-Petersson metric is a CAT(0) space with ideal points coming in natural pieces, with each piece the metric product of simpler Weil-Petersson spaces.

Q 12.5. *(Brock) What is the rank of the Weil-Petersson metric? What is the rank of the mapping class group?*

The *rank* of a metric space X is the maximal n such that X admits a quasi-isometric embedding $\mathbb{R}^n \to X$. Brock-Farb show that a lower bound for W-P is $\frac{1}{2}(3g - 3 + b)$ (Yamada's result above predicts that the rank is the maximal number of disjoint subsurfaces with noncompact Teichmüller space, and the Brock-Farb result confirms this as a lower bound). For the mapping class group, a lower bound is given by a maximal curve system – the associated group of Dehn twists is embedded quasi-isometrically.

12.1 Automorphisms of free groups (Gilbert Levitt)

$\alpha : F_n \to F_n$ will denote an automorphism.

Q 12.6. *Assuming that the fixed subgroup $Fix(\alpha)$ is cyclic, find a bound on the length of a generator of $Fix(\alpha)$ in terms of the complexity of α.*

(ed. comm.: an easier version of the question would be to bound the length of the generator of $Fix(\alpha)$ in terms of the complexity of a relative train-track representative of α.)

Q 12.7. *Which α preserve an order (invariant under right translations) on F_n? If α has periodic elements it cannot preserve an order. Are there other obstructions?*

A result of Perron-Rolfsen asserts that if $\alpha_{ab} \in GL(n, \mathbb{Z})$ has all eigenvalues > 0 then α preserves an order.

Q 12.8. *Does $Out(F_n)$ $(n > 2)$ have a right orderable subgroup of finite index?*

It is a theorem of Dave Witte (Proc.A.M.S. 122 (1994), no. 2, 333–340) that $SL_n(\mathbb{Z})$ ($n > 2$) does not have a right orderable subgroup of finite index. Mapping class groups of surfaces with nonempty boundary are orderable ([Short-Wiest] following Thurston).

Q 12.9. *Can the mapping class group (or a finite index subgroup of it) of a closed surface be embedded into $Out(F_n)$? Into the mapping class group of a punctured surface?*

Q 12.10. *Do finite index subgroups of $Out(F_n)$ ($n > 2$) have property (FA), i.e. does every isometric action of such a subgroup on a simplicial tree (or even an $\mathbb{R}$-tree) have a global fixed point?*

Update: Lubotzky constructed a finite index subgroup of $Out(F_3)$ that maps onto a nonabelian free group.

Q 12.11. *(Grigorchuk) $Aut(F_n)$ acts on the space $\partial_3 F_n$ of triples of distinct ends of F_n. Denote by Y_n the compact space (Cantor set) the quotient space of $\partial_3 F_n$ by the group of inner automorphisms. Thus $Out(F_n)$ acts on Y_n. Describe the dynamics of this action; in particular the dynamics of any individual outer automorphism.*

12.2 Schottky groups in mapping class groups (Lee Mosher)

Recall that a *Schottky group* is a subgroup F of $Isom(\mathbb{H}^n)$ which is free of finite rank, discrete, consists of loxodromic elements, and every orbit is quasiconvex. Similarly, we define a Schottky subgroup of $Isom(Teich(S)) \cong MCG(S)$ where we replace "loxodromic" by "pseudoAnosov". Note that $MCG(S)$ acts discretely on $Teich(S)$, so the word "discrete" can be omitted from the definition.

Q 12.12. *Do there exist f.g. free subgroups of $MCG(S)$ consisting of identity and pseudoAnosov mapping classes which are not Schottky?*

If such a subgroup F exists, then $\pi_1(S) \rtimes F$ is not word-hyperbolic, but has no Baumslag-Solitar subgroups. Potential candidates for such F might be found as subgroups of Whittlesey's example.

Q 12.13. *Do there exist non-free pseudoAnosov subgroups?*

Q 12.14. *Is there an $Out(F_n)$ analog of the above?*

One difficulty is that the geometry of Outer Space is not well understood. It is still true (Bridson-Vogtmann) that the group of simplicial homeomorphisms of Outer Space is $Out(F_n)$. It is not clear what a good metric on Outer Space should be. For example, if α is an irreducible automorphism, then the set of train-tracks for α should be a quasi-line, and these quasi-lines for α and α^{-1} should be a uniform distance apart (in the case of MCG they coincide). Our ignorance is exemplified in the following:

Q 12.15. *Given n is there a uniform constant K such that for every irreducible automorphism $\alpha : F_n \to F_n$*

$$\frac{\log \lambda(\alpha)}{\log \lambda(\alpha^{-1})} \leq K$$

where $\lambda(f)$ denotes the growth rate of f?

($K = 1$ in the case of MCG.)

12.3 Betti numbers of finite covers (Andrew Casson)

Q 12.16. *Does every automorphism $h : F_n \to F_n$ leave invariant a finite index subgroup K such that $h_{ab} : K/K' \to K/K'$ has an eigenvalue which is a root of unity?*

Q 12.17. *Does every closed 3-manifold M which fibers over S^1 with fiber of genus ≥ 2 have a finite cover $\tilde{M}$ with $b_1(\tilde{M}) > 1$?*

Note: Q 12.17 is a “warmup” for the well-known conjecture that every aspherical 3-manifold has a finite cover with positive b_1. Q 12.17 is Q 12.16 with surface groups instead of free groups.

12.4 QI rigidity of MCG and $Out(F_n)$ (Martin Bridson)

Q 12.18. *Is $MCG(S_g) \to QI(MCG(S_g))$ an isomorphism for $g \geq 3$?*

For $g = 2$ the hyper-elliptic involution is central and has to be quotiented out from the left-hand side.

Q 12.19. *Suppose that $\phi : MCG(S_g) \to MCG(S_g)$ is a quasi-isometry. Does ϕ map maximal flats to maximal flats (“maximal flats” come from maximal rank abelian subgroups)?*

Note: By work of Ivanov, yes to #2 implies yes to #1.

Q 12.20. *Same questions for $Out(F_n)$.*

Q 12.21. *If Γ is an irreducible uniform lattice in a higher rank connected semisimple Lie group, does every homomorphism $\Gamma \to Out(F_n)$ necessarily have finite image?*

This is true for nonuniform lattices as observed by Bridson-Farb (consequence of the Kazhdan-Margulis superrigidity and the fact that solvable subgroups of $Out(F_n)$ are virtually abelian [Bestvina-Feighn-Handel]).

12.5 Linearity of mapping class groups (Joan Birman)

Q 12.22. *Is $MCG(S_{g,b,n})$ linear?*

Here g is the genus, b the number of boundary components and n the number of punctures. By the work of Bigelow and Krammer, $MCG(0,1,n)$, $MCG(0,0,n)$, and $MCG(2,0,0)$ are linear.

12.6 The Singular Braid Monoid (Joan Birman)

A *singular braid* is defined by a braid diagram as usual, except that in addition to over- and under-crossings, crossings of strands are allowed. The standard singular braids are

- braid σ_i in which strand $i+1$ goes over the i^{th} strand, and
- singular braid τ_i in which strands i and $i+1$ cross.

The set of singular braids on n strands forms a monoid SB_n generated by $\{\sigma_1^{\pm 1}, \cdots, \sigma_{n-1}^{\pm 1}, \tau_1, \cdots, \tau_{n-1}\}$. A monoid presentation of SB_n is given in terms of these generators and the following relations:

- braid relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$,
- $\tau_i\tau_j = \tau_j\tau_i$ for $|i-j| > 1$,
- $\sigma_i\tau_j = \tau_j\sigma_i$ if $i = j$ or $|i-j| > 1$,
- $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$,
- $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$.

See J. Birman: "New points of View in knot theory", Bull AMS **28**(1993), no. 2, 253-287.

The singular braid monoid is closely related to Vasiliev invariants of knots. Define a monoid homomorphism $\Phi : SB_n \to \mathbb{Z}B_n$ to the group ring of the braid group B_n by $\Phi(\sigma_i) = \sigma_i$ and $\Phi(\tau_i) = \sigma_i - \sigma_i^{-1}$.

Q 12.23. Conjecture. Φ *is injective.*

The proof that this is true for $n = 3$ is due to A.Jarai, "On the monoid of singular braids", Topology and its Applications, **96**(1999), 109-119.

It is also true on singular braids with at most two singularities (exercise).

13 Other Questions

Q 13.1. *(Kevin Whyte) Let K be a finite complex with $\pi = \pi(K)$ amenable. Is there a uniform bound to the betti numbers of finite covers of K?*

Related to the work of Andrzej Zuk on ℓ_2-cohomology. Also related to the conjecture that such π are elementary amenable.

Q 13.2. *(Kevin Whyte) Is every solvable $PD(n)$ group polycyclic?*

Q 13.3. *(Henry Glover) Does for every finite graph G the following $1{-}2{-}\infty$-conjecture hold: G is planar, a double cover of G is planar, or no finite cover is planar?*

There is a reduction (ref??) to the graph G obtained by coning off the 1-skeleton of the octahedron.

Q 13.4. *(S. Ivanov) Is there a f.p. slender group which is not polycyclic-by-finite?*

A group is slender (or Noetherian) if every subgroup is f.g. [Olshanskii] has constructed a f.g. counterexample.

Q 13.5. *(Seymour Bachmut) Is $SL_2(K)$ finitely generated for $K = \mathbb{Z}[X, X^{-1}]$ or $K = F[X, X^{-1}, Y, Y^{-1}]$ for a field F?*

Q 13.6. *Are 1-relator groups coherent?*

13.1 Hopf-Thurston Conjecture (Mike Davis)

The Hopf-Thurston Conjecture asserts that if M is a closed aspherical $2n$-manifold then $(-1)^n\chi(M) \geq 0$. The stronger Singer Conjecture asserts that the ℓ_2-homology of $\tilde{M}$ is 0 except possibly in dimension n. See the recent Davis-Okun work. A special case (using right-angled Coxeter groups) of Hopf-Thurston's Conjecture is the following:

Let L be a flag triangulation of S^{2k-1} and define

$$\chi = 1 - \frac{1}{2}f_0 + \frac{1}{4}f_1 - \frac{1}{8}f_2 + \cdots = 1 - \sum_{i=0}^{2k-1} (-1)^i \frac{1}{2^{i+1}} f_i$$

where f_i is the number of i-simplices of L.

Q 13.7. Conjecture. $(-1)^k \chi \geq 0$.

This is true for S^3 (Davis-Okun) and it is also true if L is the barycentric subdivision of another triangulation.

13.2 Word Problem (Martin Bridson)

Q 13.8. *Do there exist groups G with balanced presentation (same number of generators and relations), with $H_1(G) = 0$ and with unsolvable word problem?*

Note: The standard examples of groups with unsolvable word problem have more relations than generators. The condition $H_1(G) = 0$ is added to rule out counterexamples obtained by adding silly generators. Any other condition that rules this out is acceptable.

Q 13.9. *Is there a sequence of (perfect, of course) groups with balanced presentations among which one cannot recognize trivial groups?*

13.3 Membership Problem in semigroups (John Meakin)

Let G be a 1-relator group with the relator W a cyclically reduced word in the generators. Let P be the submonoid of G generated by all prefixes of W.

Q 13.10. *Is the membership problem for P in G decidable?*

For motivation and special cases, see [S. Ivanov - S. Margolis - J. Meakin, One relator inverse monoids and 1-relator groups, J. Pure Applied Alg.]. The paper reduces the word problem in G to the question above.