Lecture 10. Complex Projective Varieties

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 10. The category of projective varieties, which count among their objects the Grassmannians and (eventually) the flag varieties.

In Lecture 9, we gave the definition of a variety as a separated, locally affine Noetherian topological space equipped with a sheaf of regular functions. Our first task is to find a similar categorical workaround for the notion of *compact* that we gave for *Hausdorff* and then to prove that projective varieties satisfy this criterion.

Definition 10.1 An object X in a category of topological spaces with products is *proper* if it is separated and also *universally closed*, i.e. all the projections:

$$p : X \times Y \to Y$$

onto map closed sets to closed sets.

Example 10.1. $\mathbb{C}^1 = \text{mspec}(\mathbb{C}[x])$ is not proper. The hyperbola:

$$Z(xy - 1) \subset \mathbb{C} \times \mathbb{C}$$

projects to the non-closed set $\mathbb{C} - \{0\}$.

This example can easily be expanded to show:

Exercise 10.1 The **only** affine variety that is proper in the category of affine varieties is the single point mspec($\mathbb{C}$).

This is the affine variety version of the fact that a ball in $\mathbb{R}^n$ fails to be compact. So what's a projective variety? Let's start with:

Definition 10.2. An action of $\mathbb{C}^*$ on a finitely generated $\mathbb{C}$-algebra domain A is *linear* if:

(a) The scalars $\mathbb{C} \subset A$ are the subspace for the trivial character.

(b) The subspace for the tautological character $\chi(\lambda) = \lambda^{-1}$ is finite-dimensional, and denoted $A_1 \subset A$.

(c) The algebra breaks into finite dimension character spaces and is generated by A_1:

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$$

Vocabulary. An algebra A with a linear action of $\mathbb{C}^*$ is *linearized*.

Example 10.2 Let $A = \mathbb{C}[x_1, ..., x_n]$ be the polynomial ring, with:

$$\lambda(x_i) = \lambda^{-1}x_i$$

the standard "dual scaling" action. This linearizes A, with:

$$\mathbb{C}[x_1, ..., x_n]_d = \{\text{homogeneous polynomials of degree } d\}$$

and is the action on functions associated to the scaling action on $\mathbb{C}^n$:

$$\lambda(p_1, ..., p_n) = (\lambda p_1, ..., \lambda p_n)$$

Definition 10.3. An ideal $I \subset A$ inside a linearized $\mathbb{C}$-algebra A is *homogeneous* if I is $\mathbb{C}^*$-invariant as a subspace of A.

Remark. The usual correspondence holds:

$$\{\text{homogeneous ideals } I \subset A\} \leftrightarrow \{\text{linear quotient algebras } A/I\}$$

There is a single maximal homogeneous ideal, denoted by:

$$m = A_1 \oplus A_2 \oplus \cdots$$

corresponding to the linear quotient algebra $\mathbb{C}$ (trivial action) that plays the role here that A itself played in the non-linear case. Namely, all homogeneous ideals are contained in m and *maximality* is only taken among homogeneous ideals that are properly contained in m.

Let A be a linear $\mathbb{C}$-algebra domain. Then:

(a) The homogeneous field of fractions of A is:

$$\mathbb{C}(A)_0 = \left\{ \frac{F}{G} \mid F, G \in A_d \text{ for the same } d \right\}$$

(b) For any $a \in A_1$, the localized $\mathbb{C}$-algebra A_a is:

$$A_a = A[a^{-1}]_0 = \left\{ \frac{F}{a^d} \mid F \in A_d \right\} \subset \mathbb{C}(A)_0$$

Remark. Notice that:

$$\mathbb{C}(A)_0 = \mathbb{C}(A_a)$$

is the ordinary field of fractions for **every** $a \in A_1$.

Definition 10.4. $\operatorname{mproj}(A)$ is the Noetherian topological space with a sheaf of regular functions on it defined by:

(a) The points $x \in X := \operatorname{mproj}(A)$ are the maximal homogeneous ideals $m_x \subset m$.

(b) The topology on X is the homogeneous Zariski topology:

$$Z(I) = \{\text{maximal homogeneous ideals } m_x \text{ containing } I\}$$

(together with the empty set) are the closed sets of X.

(c) The domain of definition of $\phi \in \mathbb{C}(A)_0$ is:

$$U_\phi = \left\{ m_x \in A \mid \phi = \frac{F}{G} \text{ with } G \notin m_x \right\} \subset X$$

and the sheaf $\mathcal{O}_X$ is defined by: $\mathcal{O}_X(U) = \{\phi \in \mathbb{C}(A)_0 \mid U \subset U_\phi\}$

Question. In what sense is ϕ a *function* on its domain?

There are two answers to this.

(i) The "ordinary" maximal ideals in A are all of the form:

$$m_p = \langle a_1 - p_1, ..., a_n - p_n \rangle$$

as $a_i \in A_1$ range over a basis, by the Nullstellensatz, and the maximal homogeneous ideals are all of the form:

$$m_{[p]} = \langle a_i p_j - a_j p_i \rangle$$

where $[p] = (p_1 : ... : p_n)$ is the equivalence class of p under scaling. These are the largest homogeneous ideals contained in m_p. Then:

$$f : \text{mspec}(A) - \{m\} \to \text{mproj}(A);\ f(m_p) = m_{[p]}$$

is a well-defined map, the fibers $f^{-1}(m_{[p]})$ consist of $m_{\lambda \cdot p}$ for all scalings of p, and we can evaluate a rational function ϕ at a point m_p of mspec(A) as usual, and notice that because $\phi \in \mathbb{C}(A)_0$, it is invariant under scalings and well-defined as a function on mproj(A).

(ii) Given a nonzero element $a = \sum c_i a_i \in A_1$, consider the map:

$$f_a : \text{mspec}(A_a) \to \text{mproj}(A)$$

defined as follows. The algebra A_a is generated by $a_1/a, ..., a_n/a$, so by the Nullstellensatz (again), its maximal ideals are all of the form $m_p = \langle a_i/a - p_i \rangle$ subject to the constraint that $\sum c_i p_i = 1$. The map f_a is then defined by:

$$f_a(m_p) = \langle a_i - p_i a \rangle = m_{[p]}$$

and *this map is injective* and induces an **isomorphism**:

$$\text{mspec}(A_a) \cong U_a \subset \text{mproj}(A)$$

between mspec(A_a) and the open subset $U_a = \text{mproj}(A) - Z(\langle a \rangle)$ with the Zariski topology and sheaf of regular functions on U_a inherited from that on mproj(A).

Examples 10.3. (a) For $A = \mathbb{C}[x_1, ..., x_n]$ with the scaling action from Example 10.2, the map:

$$f : \mathbb{C}^n - \{0\} = \text{mspec}(A) - \{m_0\} \to \text{mproj}(A) = \mathbb{P}^n_{\mathbb{C}}$$

is the standard quotient by the scaling action, exhibiting $\mathbb{P}^n_{\mathbb{C}}$ as the space of lines (through the origin) in $\mathbb{C}^n$. The local isomorphisms:

$$f_i = f_{x_i} : \mathbb{C}^{n-1} = \text{mspec}(\mathbb{C}[x_1/x_i, ..., x_n/x_i]) \to U_i \subset \mathbb{P}^n_{\mathbb{C}}$$

are precisely the open cover of $\mathbb{P}^n_{\mathbb{C}}$ explained in Lecture 6 by gluing vector spaces to create projective space as a manifold.

(b) If A is any linear $\mathbb{C}$-algebra, consider the natural surjective map:

$$h : \mathbb{C}[A_1] \to A$$

from the polynomial ring onto A. Then we obtain a map:

$$h^* : \mathrm{mproj}(A) \to \mathbb{P}(A_1) := \mathrm{mproj}(\mathbb{C}[A_1])$$

in the opposite direction, which is an isomorphism from $\mathrm{mproj}(A)$ to the irreducible **Zariski closed** subset of projective space $\mathbb{P}(A_1)$ defined by the homogeneous polynomial relations (in A) on the generators of A_1. This is the projective variety analogue of the closed embedding of affine varieties associated to a surjective map of $\mathbb{C}$-algebras.

(c) Fix $m > 0$ and consider the sub-algebra:

$$A = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1, ..., x_n]_{md} \subset \mathbb{C}[x_1, ..., x_n]$$

created from only the homogeneous polynomials for characters that are multiples of χ_{-m}. We can **factor** the action of $\mathbb{C}^*$ through $\lambda \mapsto \lambda^m$ to **linearize** this algebra. This creates a new grading on A with:

$$A_1 = \langle \text{all monomials of degree } m \text{ in the } x_i \rangle$$

with lots of relations among the polynomials in the elements of A_1. In fact, these are generated by quadratic relations.

The new projective variety $\mathrm{mproj}(A)$ (for the modified grading) is isomorphic *as a variety* to $\mathbb{P}^n_{\mathbb{C}}$, but its natural habitat comes from its closed embedding in $\mathbb{P}(A_1)$, as in (b).

(d) Tensor two $\mathbb{C}$-algebras A and B and consider:

$$A \otimes_{\mathbb{C}} B \text{ with the product linearization}$$

Peel off the sub-algebra (with $\mathbb{C}^*$-action):

$$\mathbb{C} \oplus (A_1 \otimes B_1) \oplus (A_2 \otimes B_2) \oplus ...$$

and rescale the linearization as in (b) to get a linearized algebra C. This turns out to be the product of $\mathrm{mproj}(A)$ and $\mathrm{mproj}(B)$.

(e) Consider the subalgebra of the polynomials in the Grassmann coordinates given by the Plücker determinants:

$$\mathbb{C}[\det(x_I)] \subset \mathbb{C}[x_{11}, ..., x_{mn}]$$

and again rescale to linearize the subalgebra. Then:

$$P = \mathrm{mproj}(\mathbb{C}[\det(x_I)]) \subset \mathbb{P}(P_1)$$

is the Plücker embedding of the Grassmannian!

So onward with the projective varieties.

Proposition 10.1. $\mathrm{mproj}(A)$ is separated and proper.