Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

1.4. Vector Bundles on Curves Part Two. We tackle the construction of the moduli spaces of semi-stable vector bundles on a non-singular projective curve.

- C is a non-singular projective curve of genus g
- L is a very ample line bundle on C and
- E is a semi-stable vector bundle on C of rank r and degree δ.
- E_S is a family of such vector bundles over a scheme S

By Riemann-Roch, the Hilbert polynomial of E with respect to L is:

$$Q(d) = \chi(C, E \otimes L^d) = d\deg(L)r + (r(1-g) + \delta)$$

We will freely confuse E with i_*E for the embedding $i : C \to \mathbb{P}^n$ determined by the very ample line bundle L. We will say, for example, that E is d-regular if i_*E is a d-regular coherent sheaf on $\mathbb{P}^n$.

Boundedness Lemma.

$$\mathrm{H}^1(C, E \otimes L^{\otimes d}) = 0$$

for all d such that $\mu(E \otimes L^d) > 2g - 2$.

Proof. This is the same as the proof for line bundles. By Serre duality,

$$\mathrm{H}^1(C, E \otimes L^{\otimes d}) = \mathrm{H}^0(C, (E \otimes L^{\otimes d})^* \otimes \omega_C)^*$$

and the inequality on d is equivalent to the slope of $(E \otimes L^{\otimes d})^* \otimes \omega_C$ being negative. A semi-stable vector bundle of negative slope has no nonzero global sections. $\square$

Corollary. Choosing d as above, we conclude that E is $d+1$-regular and so

$$E \otimes L^{\otimes d+1} \text{ is generated by global sections}$$

Moreover:

(i) For the family E_S of semi-stable vector bundles,

$$\pi^*\pi_*(E_S \otimes L^{d+1}) \otimes L^{-d-1} \to E_S$$

is surjective, where $\pi : C_S \to S$ and L is the constant (relatively ample) line bundle.

(ii) Let V be a fixed vector space of dimension $Q(d)$. Then the universal quotient:

$$V_{C_{\mathrm{Quot}}} \otimes L^{-d-1} \to \mathcal{E}$$

over the scheme $\mathrm{Quot}(C, V \otimes L^{-d-1}, Q)$ includes every semi-stable quotient:

$$V \otimes L^{-d-1} \to E$$

factoring through an isomorphism between V and $\mathrm{H}^0(C, E \otimes L^{d+1})$.

Remark. There are many other quotients, including unstable quotient bundles and quotient coherent sheaves with torsion, as well as surjective maps to semi-stable bundles such that the induced map from V to $\mathrm{H}^0(C, E \otimes L^{d+1})$ is not an isomorphism (even though the two vector spaces have the same dimension).

Example. Let $E = \mathcal{O}_{\mathbb{P}^1}$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Then we may take $d = 0$ and consider:

$$\mathrm{Quot}(\mathbb{P}^1, V(-1), d+1) \text{ for } V = k \oplus k$$

This contains quotients E of the following two types:

$$V(-1) \to \mathcal{O}_{\mathbb{P}^1} \text{ and } V(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_p$$

always (in this case) inducing isomorphisms between V and $\mathrm{H}^0(\mathbb{P}^1, E(1))$.

The set of $\mathcal{O}_{\mathbb{P}^1}$ quotients is isomorphic to:

$$\mathrm{Iso}(V, \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))/k^* = \mathrm{PGL}(2, k)$$

where $k^* = \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1})$. Similarly, each $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_p$ quotient appears as:

$$\mathrm{Iso}(V, \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}_p))/\mathrm{Aut}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_p) = \mathbb{P}^1$$

so all such quotients (for $p \in \mathbb{P}^1$) make up the complement of $\mathrm{PGL}(2, k)$ in the Quot scheme, which is isomorphic to $\mathbb{P}^3$.

Remark. The Quot schemes

$$\mathrm{Quot}(\mathbb{P}^1, V \otimes L^{-d-1}, Q)$$

admit an action of $\mathrm{GL}(V)$ that factors through an action of $\mathrm{PGL}(V)$ that is free on the locus of quotients that are **stable** vector bundles with an induced isomorphism on global sections. Thus, this locus ought to be a principal $\mathrm{PGL}(V)$-bundle over the moduli space of stable bundles. We will now prove that it is quasi-projective.

Openness of Semi-Stability and Stability. Let $\mathcal{E}_S$ be coherent sheaf on C_S that is flat over a scheme S of finite type with Hilbert polynomial Q. Then:

(i) The points of S over which $\mathcal{E}_S = E_S$ is locally free is an open subset U_{vb}.

(ii) The points of S over which E_S is semi-stable is an open subset $U_{ss} \subset U_{vb}$.

(iii) The points of S over which E_S is stable is an open subset $U_s \subset U_{ss} \subset U_{vb}$.

Proof. In each case, we can use **relative Quot schemes** of quotients of $\mathcal{E}_S$. In the first case, $\mathcal{E}_s$ fails to be locally free if and only if there is a surjection:

$$\mathcal{E}_s \to F_s$$

with kernel isomorphic to a skyscaper sheaf $\mathcal{O}_p$. Thus, the relative Quot scheme:

$$\mathrm{Quot}(C_S, \mathcal{E}_S, Q-1) \to S$$

is proper over S, with, therefore, a closed image, consisting precisely of the points for which $\mathcal{E}_s$ is not locally free. The complement is then the open subset U_{vb}.

Similarly, consider the points $s \in U_{vb}$ for which E_s is not semi-stable. Then as we've seen, E_s has a maximal destabilizing sub-bundle:

$$0 \to F_s \to E_s \to E_s/F_s \to 0$$

The set of ranks and degrees of such sub-bundles is bounded since S has finite type so there is a finite union of relative Quot schemes:

$$\mathrm{Quot}(C_{U_{vb}}, E_S, Q-P) \to U_{vb}$$

as P ranges over the Hilbert polynomials of each maximal destabilizing subbundle. This maps onto the locus of unstable vector bundles, which is therefore closed.

The argument for the openness of U_s is completely analogous. $\square$

Going back to the Corollary, we now have:

$$U_s \subset U_{ss} \subset \mathrm{Quot}(C, V \otimes L^{-d-1}, Q)$$

for the universal quotient. Moreover, the locus of semi-stable quotients for which:

$$V \to \mathrm{H}^0(C, E_s \otimes L^{d+1})$$

is **not** an isomorphism is the locus where the map $V_{U_{ss}} \to \pi_*(E_{U_{ss}} \otimes L^{d+1})$ of locally free sheaves fails to be an isomorphism, which is a divisor in U_{ss}.

Corollary. If E_R is a vector bundle on C_R (for R a DVR), then:

(i) If $E_{k(R)}$ is stable, then E_K is stable.

(ii) If $E_{k(R)}$ is semi-stable, then E_K is semi-stable.

The valuative criterion is a sort of converse to this.

Semi-stable replacement. Each family E_K of semi-stable vector bundles has a semi-stable limit $E_{k(R)}$. If the limit is stable, then it is unique up to isomorphism.

Proof. The existence of a locally free sheaf E_R on C_R restricting to E_K, and hence of a locally free limit $E_{k(R)}$ was established in the previous section under the valuative criterion. Suppose $E_{k(R)}$ is unstable, and let:

$$0 \to F_{k(R)} \to E_{k(R)} \to F'_{k(R)} \to 0$$

be the inclusion of the maximal destabilizing sub-bundle. Then the modification:

$$0 \to E'_R \to E_R \to i_* F'_{k(R)} \to 0$$

gives rise to a new limit fitting into:

$$(*)\ 0 \to F'_{k(R)} \to E'_{k(R)} \to F_{k(R)} \to 0$$

Now suppose $F''_{k(R)} \subset E'_{k(R)}$ is the maximal destabilizing sub-bundle of $E'_{k(R)}$. Then

(a) The slope of $F''_{k(R)}$ is less than the slope of $F_{k(R)}$, or

(b) The slope of $F''_{k(R)}$ is equal to the slope of $F_{k(R)}$ and the rank is smaller, or:

(c) The slope and rank are the same, and the inclusion of $F''_{k(R)}$ splits the sequence $(*)$ via an isomorphism with $F_{k(R)}$, so that E'_R, too, admits a quotient of $F'_{k(R)}$.

In the first two cases, the new limit $E'_{k(R)}$ is less unstable than $E_{k(R)}$, and we can proceed by induction. In the last case, we may do one additional elementary modification with quotient $F'_{k(R)}$:

$$0 \to E''_R \to E'_R \to i_* F'_{k(R)} \to 0$$

and conclude that the quotient:

$$0 \to E''_R \to E_R \to \mathcal{F}' \to 0$$

is a quotient sheaf defined (and flat!) over R/m^2, restricting to the quotient $F'_{k(R)}$ over R/m. We now consider the analogous extension $(*)$ for the restriction of E''_R to $C_{k(R)}$, and apply the same analysis. Continuing in this way, we either eventually obtain a limit that satisfies (a) or (b), or else we produce a quotient sheaf that is flat over R/m^n for all n, extending the quotient $F'_{k(R)}$. But since the Quot scheme has finite type over $\mathrm{Spec}(R)$, it follows that E_R itself must have a flat quotient F'_R of the same rank and degree as F', and then E_K was unstable to begin with.

Now suppose E_R and E'_R both spread E_K with stable limits $E_{k(R)}$ and $E'_{k(R)}$. Then because these are stable vector bundles of the same slope, any morphism between them is either an isomorphism or zero. Then as in the case of line bundles, we conclude that the limit is unique. □

Remarks. (i) It is possible to have limits of stable bundles that are semi-stable but not stable (in which case the moduli space of stable bundles will not be proper). But this can only happen when the rank and degree are **not** coprime.

(ii) A limit that is semi-stable but not stable need not be unique. For example:

$$t\epsilon : 0 \to \mathcal{O}_C \to E_t \to \mathcal{O}_C \to 0$$

for $t \in k^*$ and $\epsilon \neq 0$ is a constant family of non-split bundles with a split limit.

Thus by the valuative pre-criteria, we have reason to believe that the moduli of stable vector bundles (modulo the usual equivalence class on families) will be a separated scheme, possibly quasi-projective, of finite type and if n and d are coprime, then it will be proper. With the use of extension classes, we can say more about this hypothetical scheme, e.g. find itsdimension, prove it is irreducible and get a unirationality result. We let:

$\mathfrak{Vec}^{\mathrm{st}}_C(r,d)(S) = \{E_S \text{ on } C_S \mid E_s \text{ is stable, of rank } r \text{ and degree } d \text{ for all } s \in S\}/\sim$

and

$\mathfrak{Vec}^{\mathrm{st}}_C(r,L)(S) = \{E_S \text{ on } C_S \mid E_s \text{ is stable, of rank } r \text{ with } \wedge^r E_s = L \text{ for all } s \in S\}/\sim$

so that on closed points, we have:

$$\wedge^r : \mathfrak{Vec}^{\mathrm{st}}_C(r,d)(k) \to \mathfrak{Pic}^d(C)(k)$$

with fibers equal to the sets $\mathfrak{Vec}^{\mathrm{st}}_C(r,L)(k)$. It is these fibers that we now study:

Rank Two. If d is sufficiently large, then every semi-stable rank two vector bundle E with degree d satisfies $\mathrm{H}^1(C,E) = 0$ and has a global section that vanishes nowhere, giving an exact sequence:

$$0 \to \mathcal{O}_C \to E \to L \to 0 \text{ with } L = \wedge^2 E$$

Corollary. The moduli $\mathrm{Vec}^{\mathrm{st}}_C(2,L)$ of stable rank two vector bundles of fixed determinant (if it exists) is unirational and non-singular, of dimension $3g-3$.

Proof. The space of non-zero extensions (modulo scalars):

$$\mathbb{P}^n = \mathbb{P}(\mathrm{Ext}^1(L,\mathcal{O}_C)^*)$$

admits a tautological extension (and family of rank two vector bundles) on $C \times \mathbb{P}^n$:

$$0 \to \mathcal{O}_{C_{\mathbb{P}^n}}(+1) \to E_{\mathbb{P}^n} \to L \to 0$$

and therefore a surjective rational map: $f : \mathbb{P}^n - - > \mathfrak{Vec}^{\mathrm{st}}_C(2,L)(k)$.

This shows that the moduli space, if it exists, is irreducible and unirational. Moreover, $n = d + g - 1 - 1$ by Riemann-Roch, since $\mathrm{Hom}(L,\mathcal{O}_C) = 0$ and the fibers of the map f over stable bundles E are open subsets of the projective spaces $\mathbb{P}(H^0(C,E)^*)$, of dimension $d - 2(g-1) - 1$ (Riemann-Roch). The difference is:

$$3g - 3 = \text{the dimension of the moduli space}$$

(which is only coincidentally equal to the dimension of the moduli space of curves). □

Higher Rank. We proceed with $r-1$ global sections, giving:

$$0 \to \mathcal{O}_C^{r-1} \to E \to L \to 0$$

and the parameter space for this (modulo automorphisms) is the **Grassmannian**

$$\mathrm{Gr}(r-1, n+1) = \mathrm{Gr}(r-1, \mathrm{Ext}^1(L, \mathcal{O}_C))$$

of $r-1$-planes. This is equipped with a rational surjective map (for large d):

$$f : \mathrm{Gr}(r-1, n+1) - - > \mathfrak{Vec}_C^{\mathrm{st}}(r, L)(k)$$

with fibers that are open subsets of the Grassmannians $\mathrm{Gr}(r-1, \mathrm{H}^0(C, E))$. Then:

$$\dim(\mathrm{Gr}(r-1, n+1)) = (r-1)(n-r-2) = (r-1)(d+g-1-(r-1)) \text{ and}$$
$$\dim(\mathrm{Gr}(r-1, \mathrm{H}^0(C, E)) = (r-1)(d-r(g-1)-(r-1))$$

and the difference is:

$$(r-1)(r+1)(g-1) = (r^2-1)(g-1) = \text{dimension of moduli}$$

Exercise. Prove that when $g \geq 1$, the sets of semi-stable vector bundles of each rank and degree is non-zero, and that when $g \geq 2$, the sets of stable vector bundles of each rank and degree is non-zero.

Divisors on Moduli. The claim that $\mathrm{Vec}_C^{st}(r, d)$ is a quasi-projective variety (projective if $\gcd(r, d) = 1$) requires the existence of a line bundle $\mathcal{L}$ and "enough" sections of tensor powers of $\mathcal{L}$ to conclude that $\mathcal{L}$ is ample.

Rank One. From the surjective Abel-Jacobi maps (when $d \geq 2g-1$)

$$a_d : C_d(k) \to \mathfrak{Pic}^d(C)(k)$$

with fibers $\mathbb{P}(H^0(C, L)^*) = \mathbb{P}_k^{d-g}$, we may conclude that the Picard variety $\mathrm{Pic}^d(C)$ representing the functor exists (as a Hilbert scheme of projective spaces in C_d) and that it is a non-singular projective variety of dimension g. Choosing a base point $p \in C$ yields isomorphisms

$$\mathrm{Pic}^{d-1}(C) \to \mathrm{Pic}^d(C);\ L \mapsto L(p)$$

with $\mathrm{Pic}^0(C)$ being an abelian variety (the connected component of the identity).

In particular, the image of the Abel-Jacobi map:

$$a_{g-1} : C_{g-1} \to \mathrm{Pic}^{g-1}(C)$$

is the locus of line bundles with a global section. It cannot be surjective, so there are line bundles L of degree $g-1$ on C with

$$\mathrm{H}^0(C, L) = 0 = \mathrm{H}^1(C, L) \text{ because } \chi(C, L) = 0$$

On the other hand, since $\chi(C, L') = 1$ for line bundles L' of degree g, we have:

$$a_g : C_g \to \mathrm{Pic}^g(C)$$

is surjective (Abel's Theorem), and birational since the fibers are projective spaces.

We may attach a line bundle $\mathcal{L}_{L_S}$ to any family $L_S \in \mathfrak{Pic}^{g-1}(C)(S)$ as follows:

$$\mathcal{L}_{L_S} = \det R\pi_* L_S$$

for the map $\pi : C_S \to S$ over an irreducible base scheme S.

This determinant line bundle may be computed by choosing a reduced effective divisor $D = \sum_{i=1}^{d} p_i$ of large degree on C and pushing forward the sequence:

$$0 \to L_S \to L_S(D) \to L_S(D)/L_S \to 0$$

to obtain:

$$0 \to \pi_* L_S \to \pi_* L_S(D) \xrightarrow{\phi} \pi_* L_S(D)/L_S \to R^1\pi_* L_S \to 0$$

and since $\pi_* L_S(D)$ and $\pi_* L_S(D)/L_S$ are locally free sheaves of the **same** rank d,

(i) by the projection formula, the line bundle:

$$\det(R\pi_* L_S) := \wedge^d(\pi_* L_S(D)) \otimes \wedge^d(\pi_* L_S(D)/L_S)^*$$

satisfies $\det(R\pi_* L_S) = \det(R\pi_*(L_S \otimes \pi^*\mathcal{A})$ for any line bundle $\mathcal{A}$ on S.

(ii) if ϕ has full rank at some point, then the Cartier divisor support:

$$\Theta_{L_S} := \mathrm{Supp}(R^1\pi_* L_S)$$

computes the "determinant" line bundle via: $\mathcal{O}_S(-\Theta_{L_S}) \cong \det(R\pi_* L_S)$. This happens when **some** line bundle L_s in the family has no global sections.

Corollary. The image of a_{g-1} is a (Cartier) divisor Θ on Pic^{g-1}, and:

$$\Theta_{L_S} = f^*\Theta$$

for the map $f : S \to \mathrm{Pic}^{g-1}(C)$ determined by (equiv class of) the family L_S.

Theorem (Lefschetz). The line bundle $\mathcal{L} = \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)$ satisfies:

(i) $\mathcal{L}$ has a unique global section (modulo scalars).

(ii) $\mathcal{L}^2$ is generated by global sections.

(iii) $\mathcal{L}^3$ is very ample.

Note that this line bundle (and Θ divisor) are canonical, but only on the component $\mathrm{Pic}^{g-1}(C)$. For any other component $\mathrm{Pic}^d(C)$, we may choose a line bundle L of degree $d - g - 1$ and *translate* $\Theta \mapsto \Theta + L = \{L' = A \otimes L \mid \mathrm{H}^0(C, A) \neq 0\}$ to get a non-canonical theta divisor on $\mathrm{Pic}^d(C)$. This becomes relevant when we ask for relative Theta divisors of Picard varieties on a family of curves over a base scheme.

Rank Two and More. To be continued......