Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

1.3. Vector Bundles on Curves Part One. Let C be a nonsingular projective curve of genus g over an algebraically closed field k. We aim to find a moduli space for a good class of vector bundles E of rank r and degree d over C.

First, we can look at low genus:

Genus zero (Grothendieck) Each vector bundle on $\mathbb{P}^1_k$ (the only curve of genus 0) is a direct sum of line bundles,

$$E = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_r)$$

with $d = d_1 + \cdots + d_r$, which we will always arrange so that $d_1 \geq \cdots \geq d_r$.

Remark. In particular, the only simple vector bundles on $\mathbb{P}^1_k$ are line bundles.

Genus one (Atiyah) The simple vector bundles on a curve C of genus one satisfy:

(i) The rank and degree of E are coprime, and for each such rank and degree:

(ii) There is one such bundle with each given determinant $\det(E) = \wedge^r E$.

Remark. One example to keep in mind is the family of extensions:

$$\epsilon : 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$$

varying in the g-dimensional vector space:

$$\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = \mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^0(C, \omega_C)^*$$

When $\epsilon = 0$, the sequence is split and E is trivial, but when $\epsilon \neq 0$, E is **not** isomorphic to the trivial bundle (but it is also evidently not simple). One sees this from the connecting homomorphism:

$$\delta : \mathrm{H}^0(C, \mathcal{O}_C) \to \mathrm{H}^1(C, \mathcal{O}_C) \text{ with } \delta(1) = \epsilon$$

Therefore E has only a one-dimensional space of global sections when $\epsilon \neq 0$.

In general, given two vector bundles F' and F, an extension class $\epsilon \in \mathrm{Ext}^1(F, F')$ determines a vector bundle E via the short exact sequence:

$$0 \to F' \to E \to F \to 0$$

which splits if and only if $\epsilon = 0$.

Not The Moduli Space of Vector Bundles. Fix r and d, and consider:

$$\mathfrak{Vec}(S) = \mathfrak{Vec}_C(r, d)(S) = \{\text{locally free sheaves } E_S \text{ on } C \times S\}$$

such that each $E_{k(s)}$ has rank r and degree d. This is contravariant via the pullback.

We start with the pre-check for separatedness and properness.

Valuative Criterion. Let $E_K \in \mathfrak{Vec}(K)$ for the quotient field of a DVR R. Then:

(i) E_K always spreads to a locally free sheaf $E_R \in \mathfrak{Vec}(R)$, but

(ii) The isomorphism class of $E_{k(R)}$ is uniquely determined if and only if $r = 1$.

Proof. Choose an ample line bundle L on $C_R = C \times \mathrm{Spec}(R)$ so that $E_K^* \otimes L$ is generated by global sections, where E_K^* is the dual vector bundle to E_K, and consider the vector space $V = \mathrm{H}^0(C_K, E_K^* \otimes L)$ with $E_K \subset V^*_{C_K} \otimes L$ derived from the surjection $V_{C_K} \to E_K^* \otimes L$.

This spreads to a *locally free subsheaf* $E_R \subset V^*_{C_R} \otimes L$ by the properness of the Quot scheme. This gives (i) since the subsheaf $E_{k(R)} \subset V^*_{C_{k(R)}} \otimes L$ on the central fiber curve $C_{k(R)}$ is torsion free, hence locally free.

In (i), the choice of spreading of E_K depended on an embedding $E_K \subset V^*_{C_K} \otimes L$. Suppose that E_R and E'_R are two distinct spreads of E_K to C_R. Note that:

$$C_0 := C_{k(R)} \subset C_R$$

is a Cartier divisor, defined by the equation $t = 0$ (where t generates m_R). Then:

$$E'_R(-nC_0) \subset E_R$$

spreads the identity map on E_K for all sufficiently large values of n, and there is no such inclusion for sufficiently negative values of n. Let m be chosen to be minimal supporting the spreading of the identity map on E_K. Then $E'_R(-mC_0) \subset E_R$ induces a non-zero morphism:

$$f_0 : E'_{k(R)} \otimes \mathcal{O}_{C_0}(-mC_0) \to E_{k(R)}$$

of locally free sheaves on C_0 (otherwise the identity would spread for $m+1$). Moreover, the cokernel:

$$F_{k(R)} = E_{k(R)}/E'_{k(R)}(-mC_0)$$

is locally free, as one can see from the short exact sequence of coherent sheaves on the nonsingular surface C_R:

$$0 \to E'_R(-mC_0) \to E_R \to i_*F_{k(R)} \to 0$$

Thus if $r = 1$, then $E'_R(-mC_0) \to E_R$ is an isomorphism, and the line bundles over the central fiber are isomorphic, while in higher rank, the limit is in fact never unique because of the Corollary below.

Remarks. The line bundle $\mathcal{O}_{C_0}(C_0)$ is trivial, since it is pulled back from $\mathrm{Spec}(k(R))$. This is why the tensor product $E'_R \otimes \mathcal{O}_{C_R}(-nC_0)$ does not change the isomorphism class of the vector bundle $E'_{k(R)}$ on the central fiber.

Elementary Modifications. Given a locally free sheaf E_R and a surjection:

$$E_{k(R)} \to F_{k(R)} \to 0$$

the kernel sheaf E'_R in the short exact sequence of sheaves on C_R:

$$0 \to E'_R \to E_R \to i_*F_{k(R)} \to 0$$

is locally free and the two restrictions to the central fiber fit in short exact sequences:

$$0 \to F'_{k(R)} \to E_{k(R)} \to F_{k(R)} \to 0$$

$$0 \to F_{k(R)} \to E'_{k(R)} \to F'_{k(R)} \to 0$$

with defining extension classes in $\mathrm{Ext}^1(F, F')$ and $\mathrm{Ext}^1(F', F)$, respectively.

Example. Consider the trivial bundle $E_R = \mathcal{O}_{\mathbb{P}^1_R} \oplus \mathcal{O}_{\mathbb{P}^1_R}$ and the surjection:

$$\mathcal{O}_{\mathbb{P}^1_{k(R)}} \oplus \mathcal{O}_{\mathbb{P}^1_{k(R)}} \to \mathcal{O}_{\mathbb{P}^1_{k(R)}}(1) \to 0$$

Then the restriction of E'_R to $C_0 = \mathbb{P}^1_{k(R)}$ fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_{k(R)}}(1) \to E'_{k(R)} \to \mathcal{O}_{\mathbb{P}_{k(R)}}(-1) \to 0$$

which is necessarily split. Thus, it is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{k(R)}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{k(R)}}(-1)$.

Geometric Interpretation. The projectivization $\mathbb{P}(E_R)$ is an $r+1$-dimensional non-singular scheme over $\mathrm{Spec}(R)$, that specializes:

$$\mathbb{P}(E_K) \text{ to } \mathbb{P}(E_{k(R)})$$

A quotient $E_{k(R)} \to F_{k(R)}$ corresponds to a sub-projective bundle on $C_{k(R)}$:

$$\mathbb{P}(E_{k(R)}) \subset \mathbb{P}(E_{k(R)}) \subset \mathbb{P}(E_R)$$

which we may blow up in $\mathbb{P}(E_R)$. The strict transform of $\mathbb{P}(E_{k(R)})$ is the exceptional divisor for the contraction to a new projective bundle over C_R, which is $\mathbb{P}(E'_R)$. In this way, for example, trivial families of Hirzebruch surfaces $(\mathbb{F}_n)_R$ maybe modified to specialize to Hirzebruch surface $(\mathbb{F}_{n+2})_{k(R)}$ at the central fiber.

We generalize this to the following Corollary.

Corollary. A trivial family E_R (i.e. the pull back of a bundle E_k) of rank ≥ 2 can always be modified to specialize to a vector bundle that is not the pull-back of E_k.

Proof. Any vector bundle on C_0 admits surjective maps to an ample line bundle:

$$E_{k(R)} \to L_{k(R)} \to 0$$

such that there are no non-zero maps $L_{k(R)} \to E_{k(R)}$. Thus as in the example above, the elementary modification E'_R of the trivial family E_R by the surjection to $L_{k(R)}$ specializes to a bundle that is not the pull back from E_k. □

Boundedness also is a problem for vector bundles of rank ≥ 2.

Boundedness for Line Bundles. The Abel-Jacobi map:

$$\mathrm{Hilb}(C,d)(k) = C_d \to \mathfrak{Vec}(C,1,d)(k)$$

is surjective on closed points for $d \geq g$ by a Theorem of Riemann. Moreover,

$$L \mapsto L \otimes \mathcal{O}_C(p)$$

determines a bijection:

$$\mathfrak{Vec}(C,1,d)(k) \leftrightarrow \mathfrak{Vec}(C,1,d+1)(k)$$

so all the moduli functors for rank one and degree d are bounded.

Unboundedness in General. Fix $r > 1$ and d. Then direct sums:

$$E = L_1 \oplus \cdots \oplus L_r$$

of line bundles of degrees $d_1 \geq \cdots \geq d_r$ with $\sum d_i = d$ are not a bounded family. For example, there is no ample line bundle L such that $E \otimes L$ is generated by global sections for all such E (since d_r can be arbitrarily negative).

As noted in §0.1, families of line bundles still have the "isomorphism problem." But there is a projective moduli space of line bundles with a universal family of line bundles via the following adjustment:

The Picard Functor. The functor:

$$\mathfrak{Pic}^d(C)(S) = \{\text{invertible sheaves } \mathcal{L}_S \text{ of relative degree } d \text{ on } C_S = C \times_k S\}/\sim$$

modulo the equivalence relation:

$$\mathcal{L}_S \cong \mathcal{L}_S \otimes f^*\mathcal{A} \text{ for } f : C_S \to S$$

is represented by a projective moduli space $\mathrm{Pic}^d(C)$, in the sense that:

(a) Each family $\mathcal{L}_S$ determines a morphism:

$$a_{\mathcal{L}} : S \to \text{ Pic}^d(C)$$

and equivalent familes determine the same morphism.

(b) The induced map on closed points: $\mathfrak{P}ic^d(C)(k) \to \text{Pic}^d(C)(k)$ is a bijection.

(c) There is a *Poincaré line bundle* $\mathcal{P}$ on $C \times \text{Pic}^d(C)$ that pull-backs in (a) to a line bundle equivalent to each family $\mathcal{L}_S$ under the map $p^* a_{\mathcal{L}} : C \times S \to C \times \text{Pic}^d(C)$.

Note. The Poincaré line bundle is only determined up to equivalence.

Before we prove this, we address the issues with vector bundles of higher rank.

Definition. (a) The (Mumford) slope of a vector bundle E is the ratio:

$$\mu(E) = \frac{\deg(E)}{\text{rk}(E)}$$

(b) A vector bundle E on C is unstable if there is a subbundle $F \subset E$ such that

$$\mu(F) > \mu(E)$$

otherwise it is semi-stable, and **stable** if $\mu(F) < \mu(E)$ for all sub-bundles $F \subset E$.

Remark. If $F \subset E$ is injective as a map of coherent sheaves but the quotient E/F fails to be locally free, then F can be "saturated" to a subbundle $F \subset F' \subset E$ where $F' = \ker(F \to (F/E)/(F/E)_{\text{tors}})$. If different from F, the saturation has *larger* slope than the slope of F. It follows that we lose no generality above if we assume that the subbundles $F \subset E$ are saturated, so that E/F is also locally free.

Examples. (a) Line bundles are stable.

(b) Direct sums of two or more line bundles are semistable only if the line bundles have the same degree. Direct sums of two or more vector bundles are never stable.

(c) If $0 \to F' \to E \to F \to 0$ is an exact sequence of vector bundles, then:

$$\mu(F') < \mu(E) \Leftrightarrow \mu(E) < \mu(F) \text{ and } \mu(F') > \mu(E) \Leftrightarrow \mu(E) > \mu(F)$$

Schur's Lemma. If $\mu(E) \geq \mu(E')$ and E and E' are stable, then either:

(i) $\text{Hom}(E, E') = 0$ or else

(ii) E is isomorphic to E' and $\text{Hom}(E, E') = k$, generated by an isomorphism. In particular, stable vector bundles are simple but unlike simple bundles, every nonzero morphism between stable bundles of the same rank and degree is an isomorphism.

Proof. This is follows from (c) above. Given $f \in \text{Hom}(E, E')$, consider:

$$F = \ker(f) \subset E \text{ and } E \to F' = \text{im}(f) \subset E'$$

Then by stability, $\mu(F) < \mu(E)$ and $\mu(E) < \mu(F') < \mu(E') \leq \mu(E)$ generates a contradiction unless either $f = 0$ or f is an isomorphism (i.e. $F' = 0$ or $F' = E'$).

If f and g are two isomorphisms, then $f - \lambda g$ drops rank at any given point $p \in C$ for a suitable value of λ (by considering the eigenvalues) so this is **not** an isomorphism and therefore $f - \lambda g = 0$ for this value of λ. This gives (ii) $\square$

Remark. This is called Schur's Lemma by analogy with the Schur Lemma for homomorphisms of irreducible representations of a group G. In fact, this is more than an analogy when we consider the theorem of Narasimhan and Seshadri.

As another application of (c), we have:

Lemma. (i) If E, E' are semi-stable and $\mu(E) > \mu(E')$, then $\mathrm{Hom}(E, E') = 0$.

(ii) The cokernel and kernel of a map of semi-stable vector bundles:

$$f : E \to E'$$

of the same slope $\mu = \mu(E) = \mu(E')$ are also both semi-stable of slope μ. Thus the semi-stable vector bundles of slope μ are an Artinian full abelian subcategory of the category of coherent sheaves, and the stable vector bundles are the simples. That is, every semi-stable bundle E has a (non-unique) composition series, known as a **Jordan-Hölder** filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

all of whose subquotients E_i/E_{i-1} are stable.

Proof. Left to the reader.

In contrast, unstable vector bundles carry a *uniquely* defined filtration:

Lemma. The **Harder-Narasimhan** filtration of an arbitrary vector bundle F:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$$

is uniquely determined by the property that:

(i) F_i/F_{i-1} is a semi-stable vector bundle for all i and

(ii) The slopes $\mu_i := \mu(F_i/F_{i-1})$ are strictly decreasing.

Proof. By induction on the rank of F. Choose an embedding $F \subset V^* \otimes_k L$ for a line bundle L. Then the slopes of the sub-bundles of F are bounded above by $\deg(L)$, and so there is a sub-bundle $E \subset F$ of maximal slope since the slopes of subbundles of F are a discrete set. Moreover, among these sub-bundles, we may choose an E of maximal rank. Then E is semi-stable, in particular, and we may use the Harder-Narasimhan filtration $F'_\bullet$ of the quotient F' to get a Harder-Narasimhan fitration of F by setting $F_1 = E$ and $F_{i+1} = \ker(F \to F'/F'_i)$ so that the F_i fit in exact sequences:

$$0 \to E \to F_{i+1} \to F'_i \to 0$$

In particular, $\mu(E) > \mu(F'_1) = \mu(F_2/E)$ (otherwise F_2 would either have larger slope than E or it would have the same slope and larger rank). The rest of the subquotients are:

$$F_{i+1}/F_i = F'_i/F'_{i-1}$$

so the result follows. □

We will need to investigate the behavior of Harder-Narasimhan filtrations of the fibers of a *family* of vector bundles, and in particular prove an upper-semi continuity result to conclude that semi-stability (and stability) is an open condition on families of vector bundles. But first, we motivate this notion of stability by investigating the case of Riemann surfaces and the relation between semi-stable vector bundles of degree zero and (analytic) complex vector bundles with constant transition functions.

A Topological Excursion. Let $k = \mathbb{C}$ and consider a unitary representation:

$$\rho : \pi_1(p, \Sigma) \to U(n)$$

of the topological fundamental group $\pi_1(p, \Sigma)$ of a non-singular projective curve C, viewed as a marked Riemann surface $p \in \Sigma$, retaining only the differentiable structure of the curve. This defines a complex vector bundle with locally constant transition functions on Σ which is therefore analytic simultaneously for **all** complex curves C, and therefore an algebraic vector bundle (in our sense) for all such curves C of the same genus, by the GAGA principle of Serre.

In the case of line bundles, we can be very explicit. The space of representations:

$$\rho : \pi_1(p, \Sigma) \to U(1)$$

is a torus T^{2g}. Note that the choice of base point is irrelevant because $U(1)$ is abelian and changing the base point conjugates ρ by an element of $U(1)$.

On the other hand, the analytic line bundles on C are classified by:

$$\mathrm{H}^1(C, \mathcal{O}_C^*)$$

which is computed from the exponential sequence of analytic sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$$

that give the short exact sequence:

$$\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C^*) \stackrel{\deg}{\to} \mathrm{H}^2(\Sigma, \mathbb{Z}) = \mathbb{Z}$$

where $\mathrm{H}^1(C, \mathbb{Z}) \subset \mathbb{C}^g$ is a lattice. The bundles arising from representations are analytic bundles of degree zero **and conversely** since the resulting map:

$$T^{2g} \to J(C) = \mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z})$$

to the Jacobian is a diffeomorphism of tori. Thus the complex structure of C (enhancing the differentiable surface Σ) provides a complex structure of $J(C)$ that enhances the differentiable torus T^{2g}, and every algebraic line bundle of degree zero on C corresponds to some $U(1)$ representation.

Narasimhan-Seshadri Theorem: Each algebraic vector bundle V_ρ associated to a $U(n)$ representation ρ of $\pi_1(p, \Sigma)$ has degree 0 and is semi-stable.

Proof. Since $\wedge^n V_\rho = V_{\wedge^n \rho}$ which is a $U(1)$-representation, the degree is zero.

The sections of V_ρ correspond to trivial sub-representations of ρ and are therefore trivial summands. In particular, V_ρ can have no section vanishing at a point of C. On the other hand, if V_ρ is an unstable vector bundle, then it admits a sub-bundle $F \subset V_\rho$ of rank m and positive degree d, and then $L = \wedge^m F \subset \wedge^m V_\rho = V_{\wedge^m \rho}$ is a sub-line bundle of positive degree. Tensoring by a suitable $U(1)$-representation, we may convert L into **any** line bundle of degree $d > 0$. Thus in particular,

$$\mathcal{O}_C(D) = L \otimes L_\chi \subset V_{\wedge^m \rho} \otimes V_\chi = V_{\wedge^m \rho \otimes \chi}$$

for any effective divisor D of degree d with a suitable $U(1)$ representation χ. Since $\deg(D) > 0$,this would yield a section of $V_{\wedge^m \rho \otimes \chi}$ that vanishes at a point of C, contradicting the nowhere-vanishing of sections. $\square$

Remark. We will see that V_ρ is stable if and only if ρ is irreducible. To do this, we need to generalize the diffeomorphism $T^{2g} \to J(C)$ to $U(n)$-representations. That is, we need a moduli space for (semi)-stable vector bundles on C.

To Do List for Part Two:

- HN filtration data stratifies the base of a family of vector bundles
- Stable limits of stable families (over a DVR) are unique, if they exist.
- Semi-stable limits of semi-stable families always exist. But....
- Semi-stable bundles (of fixed rank and degree) are bounded.
- Discussion of the construction of moduli....