Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

1.1. Quotient Functors. We start with a contravariant functor

$$\mathfrak{M} : Schemes/k \to Sets$$

that we seek to represent with a scheme $\mathcal{M}$ of finite type over k. Recall that this means identifying the S-points of the functor with those of the scheme:

$$\mathfrak{M}(S) = \mathcal{M}(S)$$

compatibly with the pull-back, which we will write on the functor side also as:

$$a^* := \mathfrak{M}(a) : \mathfrak{M}(S) \to \mathfrak{M}(T)$$

to simplify notation. We will use the following strategy.

(i) Valuative Pre-Check. If X is a scheme over k and $k \subset R$ is a DVR, then:

$$a^* : X(R) \to X(K)$$

is the restriction of $a : \mathrm{Spec}(R) \to X$ to the generic point $\mathrm{Spec}(K) \in \mathrm{Spec}(R)$ and:

(a) a^* is injective for all R if and only if X is separated, and

(b) a^* is bijective for all R if and only if X is proper

by the valuative criterion. Thus **if** $\mathfrak{M}$ is represented by a scheme $\mathcal{M}$, then:

(a) $\mathcal{M}$ is separated if and only if each $a^* : \mathfrak{M}(R) \to \mathfrak{M}(K)$ is injective.

(b) $\mathcal{M}$ is proper if and only if each $a^* : \mathfrak{M}(R) \to \mathfrak{M}(K)$ is bijective.

(ii) Boundedness Pre-Check. A morphism $f : U \to X$ of schemes of finite type over an algebraically closed field k is surjective if and only if

$$f_* : U(k) \to X(k)$$

is surjective, i.e. it is surjective on closed points. Thus a family $f \in \mathfrak{M}(U)$ over a scheme U of finite type would determine a surjective morphism to $\mathcal{M}$ if:

$$f_*(= \text{base extension of } f \text{ to closed points}) : U(k) \to \mathfrak{M}(k)$$

is surjective, i.e. the "fibers of f" account for all points of $\mathfrak{M}(k)$.

(iii) Construction of the moduli space $\mathcal{M}$ with universal family $f \in \mathfrak{M}(\mathcal{M})$.

Remark. In practice, $\mathcal{M}$ is often the orbit space of a group G acting on U, and the construction of $\mathcal{M}$ amounts to giving the space of orbits the structure of a scheme (or stack).

(iv) Use local Artinian rings A (= finite local k-algebras) to study $\mathcal{M}$. For example, the Zariski tangent space of a scheme X of finite type over k at a (closed) point $x \in X(k)$ is the inverse image ${a^*}^{-1}(x)$ for the map:

$$a^* : X(A) \to X(k) \text{ with } A = k[\epsilon]/(\epsilon^2), \text{ the ring of dual numbers}$$

In particular, this means that if $\mathfrak{M}$ is represented by $\mathcal{M}$, then the map of sets:

$$a^* : \mathfrak{M}(A) \to \mathfrak{M}(k)$$

is fibered in k-vector spaces; the Zariski tangent spaces at the k-points of $\mathcal{M}$.

In the same vein, the singularity (or non-singularity) of $\mathcal{M}$ at points $x \in \mathcal{M}(k)$ can be analyzed by considering the maps:

$$a^* : \mathfrak{M}(A) \to \mathfrak{M}(A/I)$$

as $I \subset A$ range over ideals in local Artinian rings with residue field k.

We will study the **deformation theory** of (iv) as it applies to our moduli spaces in more detail later. For now, we will focus on the constructions (i)-(iii) of the moduli spaces.

As the first example:

The Grassmannian. Recall that the functor for the Grassmannian is:

$$\mathfrak{M}(S) = \mathfrak{G}r(V, r)(S) = \{\text{locally free quotients } q_S : V \otimes_k \mathcal{O}_S \to E_S \text{ over } S\}$$

for a fixed vector space V of dimension n over k and fixed rank r of E_S.

(i) Let R be a discrete valuation ring over k with fraction field K and consider:

$$(q_K : V \otimes_k K \to E_K) \in \mathfrak{M}(K)$$

a quotient of K-vector spaces. We may let E_R be the R-module image of:

$$V \otimes_k R \to V \otimes_k K \to E_K$$

and then this is a free R-module quotient $V \otimes_k R \to E_R$ that restricts to q_K, and its sheafification then satisfies $a^* q_R = q_K$. Now let $F_R = \ker(q_R)$ and $F_K = \ker(q_K)$. Then in particular,

$$F_R \otimes_R K = F_K$$

There are other R-submodules $F \subset V \otimes_k R$ such that $F \otimes_R K = F_K \subset V \otimes_k K$ but these are all **contained in** F_R and so the quotients map (uniquely) onto E_R:

$$(V \otimes_k R)/F = E \to E_R$$

with **torsion** kernel. In other words, F_R is the unique such submodule with a *locally free* quotient, and all other extensions q of q_K factor uniquely through q_R.

Remark. Geometrically, this says that all other coherent sheaf quotients of $V \otimes_k \mathcal{O}_R$ that restrict to a given quotient of $V \otimes_k \mathcal{O}_K$ over the generic point have larger fibers over the special point (hence the rank is non-constant) but all map onto the unique locally free extension with kernel sheaf supported on the special point of $\mathrm{Spec}(R)$.

So the Grassmannian, if it exists, is separated and proper.

(ii) Let $U \subset \mathbb{A}_k^{nr}$ be the open subscheme of **surjective** $n \times r$ matrices and let

$$q_U : V \otimes_k \mathcal{O}_U \to \mathcal{O}_U^r$$

be the universal surjective matrix over U. Note that this is a fine moduli space for:

$$\mathfrak{M}(n, r)(S) = \{\text{surjective maps } \mathcal{O}_S^n \to \mathcal{O}_S^r\}$$

which can be thought of as trivial locally free quotients $V \otimes_k \mathcal{O}_S \to E_S$ **together with** a choice of isomorphism/trivialization $E \cong \mathcal{O}_S$. In particular, there is a surjection on closed points (but not, clearly, on all S-points)

$$p : U(k) \to \mathfrak{M}(k)$$

and an action of the group $\mathrm{GL}(r, k)$ on $U(k)$ that commutes with the map p. Moreover, this action is algebraic; $\sigma : U \times_k \mathrm{GL}(r, k) \to U$ is an affine morphism of nonsingular schemes of finite type over k.

(iii) The morphism:

$$\wedge^r : U \to \mathbb{P}(\wedge^r V)$$

commutes with the action of $\mathrm{GL}(r,k)$ and makes U a **principal G-bundle** over its image, the non-singular Grassmannian of indecomposable wedges:

$$\mathrm{Gr}(V,r) = \{v_1 \wedge \cdots \wedge v_r \subset \wedge^r V\}$$

(the ideal of the Grassmannian is generated by Plücker quadrics).

Note that we immediately get a bijection:

$$\mathfrak{G}r(V,r)(k) = \mathrm{Gr}(V,r)(k)$$

and the only remaining issue is the existence of a universal quotient. But the data of a principal bundle is **equivalent** to the data of a locally free sheaf, and there is a locally free quotient:

$$q_{\mathrm{Gr}} : V \otimes_k \mathcal{O}_{\mathrm{Gr}} \to E_{\mathrm{Gr}}$$

where E_{Gr} is *descended* from the trivial quotient bundle on U. Reversing this, each quotient locally free sheaf:

$$(q_S : V \otimes_k \mathcal{O}_S \to E_S) \in \mathfrak{M}(S)$$

determines a principal $\mathrm{GL}(r,k)$ bundle P over S and a diagram of principal bundles:

$$\begin{array}{ccc} P & \stackrel{f}{\to} & U \\ \downarrow & & \downarrow \\ S & \stackrel{\overline{f}}{\to} & \mathrm{Gr}(V,r) \end{array}$$

with the property that $(\overline{f})^* q_{\mathrm{Gr}} = q_S$ as desired. □

Our goal in the rest of this section and the next is to generalize to the setting:

- the point $\mathrm{Spec}(k)$ is upgraded to $\mathbb{P}^n_k$, projective n-space over k.
- the vector space V is upgraded to the trivial vector bundle $V = \mathcal{O}^N_{\mathbb{P}^n_k}$.
- the rank r is upgraded to a polynomial $P = P(d) : \mathbb{Z} \to \mathbb{Z}$ of degree $\le n$.
- local freeness of the quotient $V \otimes \mathcal{O}_S \to E_S$ is upgraded to flatness over S of:

$$q_S : \pi^* V \to \mathcal{E}_S$$

a quotient coherent sheaf $\mathcal{E}_S$ on $\mathbb{P}^N_S$ (where $\pi : \mathbb{P}^N_S \to \mathbb{P}^N_k$ is the projection).

The Quotient Functor. We will now consider the Grothendieck quotient functor:

$$\mathfrak{Q}uot(n, V, P)$$

for a trivial vector bundle $V \otimes_k \mathcal{O}_{\mathbb{P}^n_k}$.

Theorem (Grothendieck). For fixed n, V and P, the functor:

$$\mathfrak{M}(S) = \mathfrak{Q}uot(n,V,P)(S) = \{\text{flat quotients } q_S : \pi^* V \to \mathcal{E}_S\}$$

is represented by a projective scheme $\mathrm{Quot}(n,V,P)$ of finite type over k.

We start here with (i), leaving (ii) and (iii) for the next section.

(i) The Valuative Pre-Check. Consider the map:

$$a^* : \mathfrak{M}(R) \to \mathfrak{M}(K)$$

for Discrete Valuation Rings R over k with fraction field K.

Flatness is no condition over K, so all quotient sheaves $q_K : \pi^*V \to \mathcal{E}_K$ on $\mathbb{P}^n_K$ of Hilbert polynomial P are objects of $\mathfrak{M}(K)$. On the other hand, a quotient:

$$q_R : \pi^*V \to \mathcal{E}_R \text{ on } \mathbb{P}^n_R$$

is flat over Spec(R) if and only if each of its associated points maps to Spec(K). We arrange for a quotient q_R to satisfy $a^*q_R = q_K$ by reducing to an affine cover of $\mathbb{P}^n_R$ and then gluing the quotients.

For $i = 0, ..., n$, let

$$\mathbb{A}^n_R = \text{Spec}(R[y_1, ..., y_n]) \cong (U_i)_R \subset \mathbb{P}^n_R$$

be the open sets of the standard affine cover, and $\mathbb{A}^n_k \cong (U_i)_K = (U_i)_R \cap \mathbb{P}^n_K$.

Each restriction of the quotient q_K to an open affine subset $\mathbb{A}^n_K = (U_i)_K$ is the sheafification of a $K[y_1, ..., y_n]$-module $(M_i)_K$, and as in the Grassmannian case, we define modules $(M_i)_R$ as the images of the $R[y_1, ..., y_n]$-module homomorphisms:

$$R[y_1, ..., y_n]^N \to K[y_1, ..., y_n]^N \to (M_i)_K$$

Then these modules $(M_i)_R$ are flat over R and because they are the **unique** flat modules over R that restrict to $(M_i)_K$, it follows that the sheaves they define on $(U_i)_R$ patch together to the desired flat extension of the quotient q_K with constant Hilbert polynomial over Spec(R). Moreover, as in the case of the Grassmannian, it follows that any other quotient q (necessarily with a non-constant Hilbert polynomial over R) factors uniquely:

$$q_R : \pi^*V \xrightarrow{q} \mathcal{F}_R \xrightarrow{g} \mathcal{E}_R$$

so that the kernel of g is supported over the special point of Spec(R).

Remark. When $N = 1$, then $\mathcal{E}_R$ is the structure sheaf $\mathcal{O}_{Z_R}$ of a subscheme $Z_R \subset \mathbb{P}^n_R$ that is the unique closed subscheme that is flat over R and restricts to $Z_K \subset \mathbb{P}^n_K$. Any other closed subscheme $Z \subset \mathbb{P}^n_R$ that restricts to Z_K must **contain** Z_R as a closed subscheme hence its restriction to the residue field $\mathbb{P}^n_{R/m}$ must contain the restriction $Z_{R/m}$ of Z_R as a proper closed subscheme . This is the geometric content of (i) in the case $N = 1$.