Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

1.2. The Quot Scheme. The construction of the Quot scheme representing Grothendieck's functor of quotients relies first on a boundedness result, then on constructing a coherent sheaf $\mathcal{Q}$ on projective space over a Grassmannian, for which the Quot scheme is a stratum in the flattening stratification of $\mathcal{Q}$.

Boundedness. The following Lemma (due to Mumford) bounds the Quot functor.

Lemma. Fix $n, N = \dim(V)$ and a polynomial function $P : \mathbb{Z} \to \mathbb{Z}$ of degree $\le n$. Then each subsheaf of the trivial vector bundle (abusing notation):

$$\mathcal{F} \subset V_{\mathbb{P}} = V \otimes_k \mathcal{O}_{\mathbb{P}^n_k}$$

with Hilbert polynomial P is d-regular for some d depending only on n, N and P.

Proof. By induction on n, the case $n = 0$ being trivially true.

Let $\mathcal{E}$ be the quotient sheaf, so that:

$$0 \to \mathcal{F} \to V_{\mathbb{P}} \to \mathcal{E} \to 0$$

is an exact sequence. Then if $n > 0$, there is a hyperplane $H \subset \mathbb{P}^n_k$ such that:

$$0 \to \mathcal{F}|_H \to V_H \to \mathcal{E}|_H \to 0; \ \mathcal{F}|_H = \mathcal{F} \otimes \mathcal{O}_H$$

is exact, since the hyperplane need only avoid the assoociated points of $\mathcal{E}$ so that:

$$\mathcal{T}or^i(\mathcal{E} \otimes \mathcal{O}_H) = 0 \text{ for all } i > 0$$

In that case, the sequences:

$$(*)_d : \quad 0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}|_H(d) \to 0$$

are also exact for all $d \in \mathbb{Z}$, and in particular the Hilbert polynomial satisfies:

$$P(d) - P(d-1) = P_{\mathcal{F}|_H}(d)$$

so that the Hilbert polynomial of $\mathcal{F}|_H$ is determined by P.

We may assume there is a constant d_{n-1} only depending on N and P such that:

$$\mathcal{F}_H \text{ is } d_{n-1} \text{ regular (and } d \text{ regular for all } d > d_{n-1})$$

for all $\mathcal{F}$ and (general) H. From the exact sequences $(*)_d$, we then get:

$$\mathrm{H}^{i+1}(\mathbb{P}^n_k, \mathcal{F}(d-i)) = \mathrm{H}^{i+1}(\mathbb{P}^n_k, \mathcal{F}(d-i+1)) \text{ for all } i \ge 1 \text{ and}$$
$$0 \to \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(d-2)) \to \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(d-1)) \to \mathrm{H}^0(H, \mathcal{F}|_H(d-1)) \to$$
$$\to \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d-2)) \to \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d-1)) \to 0$$

are exact sequences for all $d \ge d_{n-1}$.

It follows from Serre Theorem B that:

(i) $\mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(d_{n-1} - 1 - i)) = \mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(d_{n-1} - i)) = \cdots = 0$ for all $i \ge 2$ and

(ii) $\dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d_{n-1} - 2)) \ge \dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d_{n-1} - 1))) \ge \cdots$

so in particular once $\mathrm{H}^1(\mathbb{P}^n_k . \mathcal{F}(d-1)) = 0$ for any $d \ge d_{n-1} - 1$, then $\mathcal{F}$ is d regular.

Thus, to conclude that each $\mathcal{F}$ is $d_n = d_{n-1} + c$ regular, it suffices to show:

(a) the sequence (ii) is **strictly** decreasing to zero (after the first inequality) and

(b) $\dim(\mathrm{H}^1(\mathbb{P}^n, \mathcal{F}(d_{n-1} - 1)) \le c$, depending only on N and P.

To see (a), consider that if $d \geq d_{n-1}$, then surjectivity of the restriction:

$$(\dagger)_d \quad \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d)) \to \mathrm{H}^0(H, \mathcal{F}|_H(d))$$

implies surjectivity with d replaced by $d+1$, by virtue of the commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \times \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d)) & \to & \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d+1)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(H, \mathcal{O}(1)) \times \mathrm{H}^0(H, \mathcal{F}|_H(d)) & \stackrel{m}{\to} & \mathrm{H}^0(H, \mathcal{F}|_H(d+1)) \end{array}$$

with the surjectivity of m following from the d-regularity of $\mathcal{F}|_H$.

If any inequality (after the first one) is an equality:

$$\dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d-1)) = \dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d))$$

it follows that the restriction $(\dagger)_d$ is surjective and remains surjective for all larger values of d, so all further inequalities are equalities, and as in (a):

$$\mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d-1)) = \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d)) = \cdots = 0$$

As for (b), by the vanishing of the cohomology for $i \geq 2$, we have:

$$\dim \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(d_{n-1}-1)) - \dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d_{n-1}-1)) = P(d_{n-1}-1)$$

But

$$\mathrm{H}^0((\mathbb{P}^n_k, \mathcal{F}(d_{n-1}-1)) \subset \mathrm{H}^0(\mathbb{P}^n_k, V_{\mathbb{P}}(d_{n-1}-1))$$

and so:

$$\dim \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}(d_{n-1}-1) \leq c = P(d_{n-1}-1) + N \cdot \binom{n + d_{n-1} - 1}{n}$$

is a constant of the desired form. □

Now we are ready for the:

Construction of the Quot Scheme. Consider an object q_S of:

$$\mathfrak{Q}uot(n, V, Q)(S) = \{\text{quotients } q_S : V_{\mathbb{P}_S} \to \mathcal{E}_S \text{ on } \mathbb{P}^n_S \text{ that are flat over } S\}$$

of Hilbert polynomial Q. By the previous Lemma, there is a d_n depending only on $n, N = \dim(V)$ and:

$$P(d) = N \cdot \binom{n+d}{n} - Q(d)$$

(hence only on n, N and Q) so that:

$$\mathrm{H}^i(\mathbb{P}^n_{k(s)}, \mathcal{F}|_{\mathbb{P}^n_{k(s)}}(d)) = \mathrm{H}^i(\mathbb{P}^n_{k(s)}, V_{\mathbb{P}_S}(d)|_{\mathbb{P}^n_{k(s)}} = \mathrm{H}^i(\mathbb{P}^n_{k(s)}, \mathcal{E}|_{\mathbb{P}^n_{k(s)}}(d))$$

for all $d \geq d_n$.

Then by the Cohomology and Base Change Theorem, we obtain exact sequences of locally free sheaves on S:

$$0 \to \pi_* \mathcal{F}_S(d) \to \pi_* V_{\mathbb{P}_S}(d) \to \pi_* \mathcal{F}_S(d) \to 0$$

where $\pi : \mathbb{P}^n_S \to S$ is the projection and:

$$\pi_* V_{\mathbb{P}_S}(d) = (V \otimes_k k[x_0, ..., x_n]_d)_S$$

is a trivial vector bundle. This gives us in particular a morphism to a Grassmannian:

$$f : S \to \mathrm{Gr}_d := \mathrm{Gr}(V \otimes k[x_0, ...x_n]_d, Q(d))$$

Let:

$$0 \to F \to (V \otimes_k k[x_0, ..., x_n]_d)_{\mathrm{Gr}_d} \to E \to 0$$

be the universal quotient (and kernel) on the Grassmannian, so that:

$$\pi_* \mathcal{F}_S(d) = f^* F$$

by the universal property. Then the map:

$$\pi^* \pi_* \mathcal{F}_S(d) \to \mathcal{F}_S(d)$$

is surjective (by regularity of the fibers), and so:

$$(\pi^* f^* F)(-d) \to V_{\mathbb{P}^n_S} \to \mathcal{E}_S \to 0$$

is (right) exact. But this sequence is intrinsic to the Grassmannian. Namely:

$$\pi^* F \to V \otimes \mathcal{O}_{\mathbb{P}^n_{\mathrm{Gr}}}(d) \to \mathcal{Q}(d) \to 0$$

(factoring through $\pi^*(V \otimes_k k[x_0, ..., x_n]_d \otimes \mathcal{O}_{\mathrm{Gr}})$) defines a coherent sheaf $\mathcal{Q}$ on projective space over the Grassmannian. The stratum $Z_Q \subset \mathrm{Gr}_d$ of the flattening stratification of $\mathcal{Q}$ associated to the Hilbert polynomial Q is therefore the universal locally closed subscheme such that the restriction of $\mathcal{Q}$ to $\mathbb{P}^n_{Z_Q}$ is flat with Hilbert polynomial Q. This represents the functor $\mathfrak{Quot}(n, V, Q)$! $\square$.

Remark. Thus the functor of quotients is represented by a locally closed subscheme, but because it is proper (proved in the last section), it follows that Z_Q is a closed subscheme of Gr_d, and therefore also projective, via the Plücker embedding.

More General Schemes of Quotients. We now have projective schemes that represent the functor of flat quotients of fixed Hilbert polynomial for:

- Trivial vector bundles V on projective space $\mathbb{P}^n_k$.
- (Easy generalization) Arbitrary coherent sheaves $\mathcal{V}$ on projective space $\mathbb{P}^n_k$.
- (Also easy) Arbitrary coherent sheaves $\mathcal{V}$ on X, a projective scheme equipped with an ample line bundle L, with respect to which the Hilbert polynomial of $\mathcal{E}$ is:

$$P(d) = \chi(X, \mathcal{E} \otimes L^{\otimes d})$$

- (Harder generalization). A T-scheme representing the quotient functor for a coherent sheaf $\mathcal{V}$ on a proper T-scheme $X \to T$ equipped with a relatively ample line bundle, commuting with base change (requiring $\mathcal{V}$ to be flat over T?) and thus defining a family (definitely not flat, in general) of Quot schemes.

Quot Schemes on Curves and Hilbert Schemes of Curves. We will use some specific schemes of quotients to construct moduli of semi-stable vector bundles on a non-singular curve and moduli of stable curves.

(a) Let C be a nonsingular projective curve of genus g over k. Any line bundle:

$$L = \mathcal{O}_C(D) \text{ of positive degree}$$

is ample, by the Riemann-Roch Theorem. In fact, the line bundles:

$$L^{\otimes d} \text{ are very ample for all } d \geq 2g + 1$$

so this power is uniform among all curves of genus g, and if we instead consider:

$$\omega_C \otimes L^d$$

then this is very ample for all $d \geq 3$, which is uniform for **all** curves.

A vector bundle E on C has rank r and degree $\delta = \deg(\wedge^r E)$. Then for any line bundle L of degree one, we have:

$$\chi(C, E \otimes L^{\otimes d}) = rd + (r(1-g) + \delta)$$

is the Hilbert polynomial of E, which only depends upon r and δ. Now consider:

$$\text{Quot}(C, V_C, r, \delta)$$

the scheme of flat quotients of the trivial rank n vector bundle V_C. The **kernel** of such a quotient is a locally free subsheaf $F \subset V_C$ of rank $n - r$ and degree $-\delta$.

Consider the case $n = 1$ and $r = 0$. This is the Hilbert scheme:

$$\text{Hilb}(C, \delta)$$

of (flat families of) subschemes $Z \subset C$ of length δ. Note that the union of diagonals:

$$\mathcal{Z}_{C^\delta} = \cup_{i=1}^{\delta} \Delta_{0,i} \subset C \times C^d$$

is flat over C^δ, and therefore defines a morphism:

$$C^\delta \to C_\delta := \text{Hilb}(C, \delta)$$

that commutes with the action of the symmetric group Σ_δ on C^δ. We will see that the Hilbert scheme is non-singular, and conclude that C_δ is the quotient of C^δ by the action of the symmetric group.

Remark. If C is replaced by a nonsingular variety X of larger dimension, then the union of diagonals in $X \times X^\delta$ is **not** flat over X^δ, and therefore does not determine a morphism to the Hilbert scheme!

On the other hand, looking at the back end of the quotient, we have:

$$\mathcal{O}_C(-D) \subset \mathcal{O}_C$$

for each (Cartier!) subscheme $D \in C_\delta$ and each line bundle subsheaf L of $\mathcal{O}_C$ appears once for each non-zero section $s \in \text{H}^0(C, L^*)$ modulo the action of the automorphism group k^* of L (which fixes L as a subsheaf of $\mathcal{O}_C$). Once we establish the existence of the Picard group of line bundles, this will give the Abel-Jacobi map:

$$C_\delta \to \text{Pic}^{-\delta}(C) = \text{Pic}^\delta(C) \text{ with fibers } |D| = \mathbb{P}(\text{H}^0(C, \mathcal{O}_C(D))^*)$$

taking the effective Cartier divisor D to the line bundle $\mathcal{O}_C(D)$.

This has an interesting generalization to Weil's "generalized" symmetric product:

$$\text{Quot}(C, V_C, 0, \delta)$$

These are also smooth, with a "determinant" map:

$$(F \to V_C \to \mathcal{E}) \to (\wedge^n F \subset \wedge^n V_C = \mathcal{O}_C \to \mathcal{O}_Z)$$

where $\mathcal{E}$ is a length-d quotient of the trivial bundle V. This gives a morphism:

$$\text{Quot}(C, V_C, 0, \delta) \to C_\delta$$

but when we look at the "other side," the situation is more complicated. Each rank n vector sub-bundle $F \subset V_C$ appears in a locally closed but not (usually) closed subset $U_F \subset \text{Quot}(C, V_C, 0, \delta)$ corresponding to collections of n sections of F^* that generically span F^* modulo the action of the automorphism group of F.

If there were a moduli space for all vector bundles of rank n and degree δ, analogous to the Picard variety, then these locally closed subsets U_F would be the fibers of a morphism from the Quot scheme, and would therefore be closed sets. The fact that they are not closed means there is no such moduli space.

Note that when $r > 0$, we get an open subset:

$$U \subset \mathrm{Quot}(C, V_C, r, \delta)$$

defined as the largest subset over which the universal quotient sheaf $\mathcal{E}_{\mathrm{Quot}\times C}$ is free. The points of U therefore parametrize quotient locally free sheaves:

$$V_C \to E \to 0$$

of rank r and degree δ, i.e. morphisms:

$$f : C \to \mathrm{Gr}(V, r)$$

of a fixed degree determined by δ. Thus the Quot scheme can be thought of as a *compactification* of the space of morphisms from C to the Grassmannian.

(b) The Hilbert schemes of "curves" $Z \subset \mathbb{P}^n_k$ are:

$$\mathrm{Hilb}(\mathbb{P}^n_k, Q) = \mathrm{Quot}(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}, Q)$$

where:

$$Q(d) = \delta \cdot d + 1 - p_a$$

and δ is the degree of $Z \subset \mathbb{P}^n_k$ and p_a is the arithmetic genus of Z.

When $n = 2$, such a subscheme $Z \subset \mathbb{P}^2_k$ consists of a Cartier divisor of degree δ and a finite residual scheme to make up the difference between:

$$1 - p_a \text{ and } 1 - \binom{d-1}{2}$$

the latter being the constant term for the Cartier divisor, which is minimal among the constants for which the Hilbert scheme is non-empty. For higher values of n, the situation is more complicated, but two cases are of particular interest:

Canonical Curves. When $n = g - 1$, $\delta = 2g - 2$ and $g = p_a$, then each smooth:

$$C \subset \mathbb{P}^{g-1}$$

is a non-hyperelliptic curve embedded by the canonical linear series, and a non-singular point of the Hilbert scheme. These curves form an irreducible open subset of the Hilbert scheme, of dimension:

$$3g - 3 + \mathrm{dimPGL}(g)$$

(as we will see in §2), allowing us to conclude that the dimension of the moduli of curves of genus g is $3g - 3$.

Large Degree Curves. For $\delta = g + n$ and $g = p_a$, the curves $C \subset \mathbb{P}^{\delta - g}$ embedded by complete linear series of divisors of degree δ are an irreducible open subset that is again non-singular of dimension:

$$3g - 3 + g + \dim \mathrm{PGL}(n + 1)$$

(again, we will see this in §2) which in this case has g extra dimensions, accounting for the choice of a line bundle in $\mathrm{Pic}^\delta(C)$.

The Idea. To interpret these open subsets of the Hilbert scheme as principal PGL-bundles over the moduli of (non-hyperelliptic) curves and the universal Picard bundle over the moduli of non-singular Riemann surfaces, respectively.

Remark. There are many other irreducible components to these Hilbert schemes. One component is analogous to the $n = 2$ case, namely, that of a Cartier divisor of degree δ in a plane $\Lambda \subset \mathbb{P}^3_k$, together with a residual scheme of (many!) points. There is interesting behavior already for the Hilbert schemes of many points in $\mathbb{P}^3$, but one component of that Hilbert scheme is the closure of the irreducible open subset consisting of distinct points. The plane curves plus distinct points also give a non-singular component of the Hilbert schemes above, though of much larger dimension and of no interest for the construction of moduli spaces.