Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

0.1. Introduction to Moduli Spaces. **Moduli** in algebraic geometry refers to the continuous variation of geometric objects that are bundled in families, and a **moduli space** is a universal base over which such families are sewn together. This is roughly analogous to the classifying space BG for a topological group G, which is the base of a universal (and contractible) principal G-bundle $EG \to BG$. This sews together all principal G-bundles, in the sense that each such G-bundle $P \to X$ is pulled back from the universal bundle via:

$$\begin{array}{ccc} P & \stackrel{\tilde{f}}{\to} & EG \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\to} & BG \end{array}$$

where f is a continuous map (well-defined up to homotopy).

The projective space $\mathbb{P}(V)$ of variable line quotients $q : V \to L$ of a fixed vector space V over a field k is a particularly simple and satisfying example of a moduli space in algebraic geometry, in which line quotients are bundled into families over a base scheme S in quotients of the form $q : V \otimes_k \mathcal{O}_S \to \mathcal{L}$, where $\mathcal{L}$ is a line **bundle** and q is a surjective "framing" of $\mathcal{L}$ by the trivial vector bundle. All such families are sewn into *the* universal quotient $(*)\ V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1)$ over the base $\mathbb{P}(V)$, which is the (fine) moduli space for bundles of line quotients. In other words, the data of a family of quotients q is in a natural bijection with the morphisms $f : S \to \mathbb{P}(V)$.

This generalizes to the Grassmannian (variable quotients of V with fixed rank) and "classical" homogeneous spaces (variable flags of quotients with fixed ranks, possibly isotropic for a fixed non-degenerate symmetric or skew-symmetric form). All are smooth projective varieties, equipped with a Plücker embedding. These are also equipped with the transitive action of a Lie group, which exhibits them as coset spaces for a (parabolic) subgroup of a Lie group, i.e. homogeneous spaces.

Grothendieck recognized in an astounding series of papers (FGA) that one could vastly generalize this to the setting of varying *coherent sheaf* quotients $q : V \to \mathcal{F}$ of a fixed coherent sheaf V on a projective variety (or scheme) X. In Grothendieck's formulation, the **Hilbert polynomial** of $\mathcal{F}$ generalizes the rank of the quotient and **flatness** is how to bundle coherent sheaf quotients with fixed Hilbert polynomial. Grothendieck's theory of schemes is beautifully suited to moduli problems, and his **Hilbert schemes** are the foundation of the modern theory of moduli.

This is particularly interesting when $V = \mathcal{O}_X$ is the "structure sheaf" of the projective scheme X, in which case each coherent-sheaf quotient $q : \mathcal{O}_X \to \mathcal{O}_Z$ is the structure sheaf of a closed subscheme $Z \subset X$ (with the given Hilbert polynomial). One instance of this is familiar. The Hilbert scheme of *hypersurfaces* $Z \subset \mathbb{P}^n$ in projective n-space is itself the (projective) space of homogeneous polynomials $F(x_0, ..., x_n)$ of degree d (which determines the Hilbert polynomial). Once one passes to codimension two or more, however, the Hilbert schemes are much more "interesting," and are typically very complicated projective schemes. Indeed, these satisfy a "Murphy's Law"; every pathology of projective schemes of finite type are on display for suitable Hilbert schemes.

If we dispense with the framing by V and consider only (flat) families of coherent sheaves $\mathcal{F}$ on a fixed projective scheme X, then we need to deal with the issue of **automorphisms**. Already when X is a point scheme (associated to a field k), the coherent sheaves are finite-dimensional vector spaces over k, and the automorphism group of $\mathcal{F} = V$ is the general linear group $\mathrm{G} = \mathrm{GL}(V, k)$. In that case, families of vector spaces over a scheme S are vector **bundles** (locally free sheaves) $\mathcal{E}$ of rank equal to the dimension of V. There is only one vector space V of a given rank up to isomorphism, so the moduli space ought to be a point, with universal family V. On the other hand, the pull-back of V from a constant map is:

$$V \otimes_k \mathcal{O}_S$$

the trivial vector bundle, and **not all vector bundles are trivial**, though they are all *locally* trivial. This is a rather serious problem, which relates back to the example at the very beginning. In this case, the moduli space is the classifying space (stack) BG, which, in algebraic geometry, is a groupoid: a category with a single object and arrows corresponding to the elements of the group G. In a very rough sense, a stack in algebraic geometry is an algebraic gadget whose "points" are groupoids with arrows constituting the inertia group of the "point", which are generally dependent upon the point.

When L is a line, there is a fix that allows the bare point to be the moduli space. Line bundles are invertible sheaves, so we can introduce an equivalence relation on families of lines (line bundles) over a scheme S:

$$\mathcal{L} \sim \mathcal{L} \otimes \mathcal{A} \text{ for all line bundles } \mathcal{A}$$

This is clearly an equivalence relation, but in fact it has only one equivalence class: every pair of line bundles on S is equivalent for this relation! Thus *up to equivalence* the point with L is the universal family.

This cheap trick becomes much more interesting when we consider families of line bundles on a variety X. Line bundles are themselves bundled into families over a base S as line bundles (invertible sheaves) $\mathcal{L}$ on the **product**:

$$X \times S \text{ with projections } p : X \times S \to S \text{ and } q : X \times S \to S$$

In this context, the equivalence relation that one takes is:

$$\mathcal{L} \sim \mathcal{L} \otimes q^*\mathcal{A} \text{ for the } \textit{pull-back} \text{ of a line bundle } \mathcal{A} \text{ on } S$$

With this equivalence relation, the moduli space of line bundles on X is the **Picard group** $\mathrm{Pic}(X)$ and a universal line bundle on $X \times \mathrm{Pic}(X)$ (well-defined up to the equivalence relation!) is called "the" **Poincaré line bundle**. In fact, the Picard variety is a group (with tensor product) and the **Jacobian** is the connected component $\mathrm{Pic}^0(X)$ of the identity line bundle $\mathcal{O}_X$. This is an abelian variety.

When we pass to vector spaces the trick doesn't work, since one cannot "undo" a vector bundle by tensoring with another vector bundle. However, there is an open class of **simple** vector bundles E on X whose isomorphism group is $\mathrm{GL}(1, k)$, just as is the case for lines and line bundles. In this case, the equivalence relation:

$$\mathcal{E} \sim \mathcal{E} \otimes q^*\mathcal{A}$$

is adequate to have a moduli space of simple vector bundles on X. Unfortunately, this space is usually not Hausdorff (i.e. not separated)! To remedy this, we restrict further to **stable** vector bundles, which are analogous to irreducible representations.

When $X = C$ is a curve (smooth, projective, of genus ≥ 2), we have the following

Theorem (Narasimhan, Seshadri). The moduli space of stable vector bundles of rank r and degree d on C is smooth and quasi-projective, and projective when r and d are coprime. When $d = 0$, there is a bijection (even a diffeormorphism) between the space of irreducible unitary representations of $\pi_1(C)$ and the moduli space of stable vector bundles that can be extended to reducible representations (and polystable vector bundles).

We will study these moduli spaces and some of their **intersection theory**.

Each projective scheme X of finite type over a field k carries a **Chow Group** and an **Intersection Ring**. The elements of the Chow group (graded by dimension) are formal sums of closed subvarieties of X modulo rational equivalence (analogous to homological equivalence of chains of simplices in singular homology) and the elements of the intersection ring are the **Chern classes** of vector bundles on X, graded by codimension, which operate on the Chow group much as cohomology classes operate on homology classes. This enables one to do enumerative geometry on (projective) moduli spaces. Applications of intersection theory include the Riemann-Roch Theorems computing Euler characteristics of coherent sheaves on smooth, projective varieties X, generalized by Grothendieck to the setting of smooth, projective **morphisms** of schemes.

We will take up Fulton's treatment of intersection theory, with a specific view towards making computations analogous to the **Schubert Calculus** enumerative geometry of the Grassmannian $\mathrm{Gr}(k^n, r)$ of r-dimensional quotients of k^n. This smooth, projective variety of dimension $r(n-r)$ is equipped with the universal quotient vector bundle:

$$(*) \quad \mathcal{O}^n_{Gr(V,r)} \to \mathcal{E}$$

The intersection ring of the Grassmannian is isomorphic to:

$$I(n,r) = \mathbb{Z}[\sigma_1, ..., \sigma_r]/\langle \frac{\partial W}{\partial \sigma_i} \rangle$$

where the σ_i are the elementary symmetric polynomials in $x_1, ..., x_r$ and:

$$W = \frac{x_1^{n+1}}{n+1} + \cdots + \frac{x_r^{n+1}}{n+1}$$

is a polynomial in $\sigma_1, ..., \sigma_r$ since it is symmetric in $x_1, ..., x_r$. The isomorphism is achieved by setting $\sigma_i = c_i(\mathcal{E})$, the **Chern classes** of the universal quotient bundle. In this way the universal quotient directly accounts for all the intersection theory of the associated moduli space. Moreover, the Chow group is a free abelian group isomorphic to $I(n, r)$ with a basis of Schubert cycles given by explicit closed subvarieties of the Grassmannian. This allows one to directly intersect elements of the Chow group, and yields the following positivity result:

Littlewood-Richardson. Every product of Schubert cycles is a sum of Schubert cycles with positive integer coefficients determined by a precise combinatorial rule (on Young tableaux).

Enumerative problems do not set up this nicely on moduli spaces in general. That's the fun of it!

Smooth projective morphisms (in the context of the G-R-R Theorem) are also how one bundles smooth projective varieties into families over a base scheme. When a family of non-singular complex projective varieties bundled into a smooth projective morphism

$$f : X \to S$$

Each such morphism is a submersion of differential manifolds, and by a theorem of Ehresmann, the map f is (diffeomorphically) a local fibration. The moduli of the fibers can therefore be thought of locally on S (in the Euclidean topology) as a continuously varying family of complex projective manifolds that share the same underlying differentiable manifold. This is the setting of **Hodge Theory**.

Any invariant of geometric objects that is constant in families over a *connected* base is dubbed a discrete invariant. Moduli is thus a feature of families of geometric objects with fixed discrete invariants. In the context of moduli of vector bundles on a curve, the **rank** and **degree** of the bundles are the discrete invariants, generalized in higher dimension to the images of Chern classes in singular cohomology(!).

Dimension is such an invariant for smooth projective morphisms and among the families of **curves** (Riemann surfaces) $C \to S$ (dimension one), the *genus* of a curve is the only other discrete invariant. Recall that the genus of a complex curve is half the first betti number, which is discrete by the fibration theorem, for example. For families of projective manifolds of higher dimension, there are more discrete invariants (e.g. more betti numbers).

Rough Definition of $\mathfrak{M}_g$. The moduli space of Riemann surfaces of genus $g \geq 2$ is the base of a universal family of complex projective curves of genus g.

(Rhetorical) Question. Why restrict to genus 2 or more?

Genus Zero. The Riemann sphere $\mathbb{CP}^1$ is the only Riemann surface of genus zero, so in a sense the moduli problem is easy to solve: the moduli space of genus zero curves is a point and the universal family is $\mathbb{P}^1$. On the other hand, this does not capture the non-trivial *families* of genus zero curves.

The *automorphism group* $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$ is responsible for bundles of projective lines that are locally trivial but globally non-trivial, just as we saw in the case of families of vector spaces.

A trick to fix the automorphism problem in this context is to rigidify the moduli problem by adding more information. In the case of genus zero curves, this can be done by choosing three distinct points $p_1, p_2, p_3 \in C$. Every such marked curve is *uniquely* isomorphic to the Riemann sphere $\mathbb{CP}^1$ with chosen points $0, 1, \infty$. In this case the moduli space is again a point, and *every* family $C \to S$ of genus zero curves equipped with three distinct section $\sigma_1, \sigma_2, \sigma_3 : S \to C$ is uniquely isomorphic to the trivial family. So in this case, the moduli space is legitimately the single point.

It is very interesting to consider the moduli spaces of genus zero curves with more points. There are several approaches to this, including applying **Geometric Invariant Theory** to collections of points on $\mathbb{P}^1$ (for the action of the automorphism group) or to go to the Deligne-Mumford compactification to handle what one does when points collide. But this is just a special case of the Deligne-Mumford approach to moduli of curves of all genus together with marked points.

Elliptic Curves. Each genus one curve is a principal homogeneous space over an elliptic curve that is a group via the choice of an origin $e \in C$. This choice "nearly" rigidifies the moduli problem, leaving only a finite group of automorphisms. One can actually tally all the automorphism groups, recalling that every elliptic curve is (analytically) isomorphic to the quotient of $\mathbb{C}$ by a *lattice* $\Lambda \subset \mathbb{C}$.

- all but two elliptic curves have $\mathrm{Aut}(C, e) = \mathbb{Z}/2\mathbb{Z}$
- the square lattice elliptic curve has $\mathrm{Aut}(C, e) = \mathbb{Z}/4\mathbb{Z}$.
- the hexagonal lattice elliptic curve has $\mathrm{Aut}(C, e) = \mathbb{Z}/6\mathbb{Z}$.

Nevertheless, we may set about looking for a universal family by considering the following family of genus one curves over $S = \mathbb{C} - \{0, 1\}$ in *Legendre normal form*:

$$f : C \subset \mathbb{CP}^2 \times S \to S \text{ defined by the equation } y^2 z = x(x - z)(x - sz)$$

This is equipped with the section $e = (0 : 1 : 0) \times S$ that picks out the origins of the fiber curves (C_s, e_s). The Ehresmann fibration theorem in this context is the observation that the fibers of this family are all (diffeomorphic) tori, and the moduli is the variation of the *cross ratios* of the sets $\{0, 1, s, \infty\}$ and $\{0, 1, \lambda, \infty\}$. These four points are the branch points of the map $C_\lambda \to \mathbb{CP}^1$; $(x : y : z) \mapsto (x : z)$ with e_λ mapping to the point at ∞.

This sets up an action of the symmetric group Σ_3 on S, generated by elements:

$$\lambda \mapsto 1/\lambda \text{ and } \lambda \mapsto \lambda/(1 - \lambda)$$

of degree two and three, respectively, whose orbits correspond to the sets of elliptic curves with the same cross ratio. This action is free away from the two orbits:

$$\lambda = 2, -1, 1/2 \text{ and } \lambda = \omega, 1/\omega$$

with stabilizers $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ that act as automorphisms on the elliptic curves, which must then correspond to the square lattice and hexagonal lattice quotients of $\mathbb{C}$! (This also shows that the analytic automorphisms of $\mathbb{C}/\Lambda$ are algebraic.) The "extra" automorphisms are generated by the involution of C_λ given by the map $y \leftrightarrow -y$ that fixes all the branch points.

In particular, the family C has non-trivial moduli and the quotient space S/Σ_3 is a variety (the j-line in the number theory literature) whose points parametrize each elliptic curve once, i.e. it is a candidate for the moduli space of elliptic curves. However, as a consequence of the presence of automorphisms, there is no universal elliptic curve $C \to S/\Sigma_3$, even if S/Σ_3 is interpreted as an orbifold (or stack) quotient of S. The actual moduli stack also accounts for the involutions.

The morals to take from this example are: (1) the moduli of curves of genus $g \geq 1$ are extremely interesting and well worth studying, but (2) one has to come to terms with the presence of finite groups of automorphisms. (Side note: Unlike the curves of genus 0 and 1, every unmarked Riemann surface of genus 2 or more has only finitely many automorphisms). The finiteness of automorphism groups means that the stack approach to moduli is much more closely tied to "ordinary" geometry. In fact, the moduli stacks of curves (and their Deligne-Mumford compactifications) are **orbifolds**.