Course Notes for Math 780-1 (Geometric Invariant Theory)

4. Vector Bundles on a Smooth Curve. In this section, we will construct projective moduli spaces for semistable vector bundles on a smooth projective curve C using GIT. The construction we present is due to Simpson, and will be generalized to higher dimensions in §5.

Let F and E be vector bundles on a smooth curve C of genus g.

Definition: (a) The *slope* $\mu(E) = \deg(E)/\mathrm{rk}(E)$.

(b) E is *stable* if $\mu(F) < \mu(E)$ for all proper subbundles $F \subset E$.

(c) E is *semistable* if $\mu(F) \le \mu(E)$ for all $F \subset E$.

Lemma 4.0: If $0 \to F \to E \to G \to 0$ is an exact sequence of vector bundles, then $\mu(F) \ge \mu(E)$ (resp.$>$) if and only if $\mu(E) \ge \mu(G)$ (resp. $>$).

Proof: Arithmetic! If $a, b, c, d > 0$, then $\frac{a}{c} > \frac{a+b}{c+d}$ if and only if $\frac{b}{d} < \frac{a+b}{c+d}$.

Examples: (i) Every vector bundle on $\mathbf{P}^1$ splits as a sum of line bundles, so there are no stable bundles on $\mathbf{P}^1$, and the only semistable ones are $\oplus\mathcal{O}(d)$.

(ii) E is stable (resp. semistable) iff E^* is stable (resp. semistable).

(iii) A line bundle L is always stable. E is stable (resp. semistable) iff $E \otimes L$ is stable (resp. semistable).

(iv) If E is semistable and:

(a) $\deg(E) < 0$, then $\mathrm{H}^0(C, E) = 0$.

(b) $\deg(E) > 2r(g-1)$, then $\mathrm{H}^1(C, E) = 0$.

(c) $\deg(E) > 2rg - r$, then E is generated by its global sections.

(Schur's) Lemma 4.1: (a) If E and F are stable with the same slope, then any map $f : E \to F$ is either 0 or an isomorphism.

(b) The only automorphisms of a stable bundle E are multiplication by scalars.

(c) If E is semistable, then there is a filtration $0 = E_0 \subset E_1 \subset ... \subset E_n = E$ such that $F_i := E_i/E_{i-1}$ is a stable vector bundle with $\mu(F_i) = \mu(E)$. Moreover, the *associated graded* $\oplus_{i=1}^n F_i$ is independent of the choice of filtration.

Proof: If $f : E \to F$ is nonzero and $\ker(f) \neq 0$, then $\ker(f)$ and $E/\ker(f)$ are bundles. Now the stability of E implies $\mu(\ker(f)) < \mu(E)$, which implies $\mu(E/\ker(f)) > \mu(E) = \mu(F)$, contradictng the stability of F. So f is injective, and surjective, by the stability of F. This gives (a).

If $\alpha : E \to E$ is an automorphism, let λ_x be an eigenvalue of the restriction of α to the fiber of E over $x \in C$. Then $\alpha - \lambda_x(\mathrm{id})$ drops rank at x, so it must be zero(!) and we have (b). Finally, (c) follows from (a) by the usual Jordan-Hölder decomposition.

(Harder-Narasimhan) Lemma 4.2: If E is any vector bundle on C, then there is a unique filtration:

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

with the property that $F_i := E_{i+1}/E_i$ are all *semistable* bundles, and $\mu(F_i) > \mu(F_{i+1})$. (This is called the Harder-Narasimhan filtration of E.)

Proof: Let $S = \{a \in \mathbf{Q} | a = \mu(F) \text{ for some quotient bundle } E \to F\}$. We claim first that S contains a minimal element. Indeed, let D be a divisor of large enough degree so that $E(D)$ is generated by its sections. Then for any quotient F, the twist $F(D)$ is also generated by its sections, hence is of nonnegative slope. Thus S is bounded from below by $-\deg(D)$, and since the elements of S have bounded denominators, it follows that S contains its lower bound, say a_{min}.

Suppose $\alpha : E \to F$ and $\alpha' : E \to F'$ are two quotients, with $\mu(F) = \mu(F') = a_{min}$. Then the image of $\alpha \oplus \alpha' : E \to F \oplus F'$ is necessarily semistable, of slope a_{min}, and surjects onto F and F'. So there is a unique quotient $\alpha_{max} : E \to F_{max}$ of maximal rank. Let $K = \ker(\alpha_{max})$. Then any quotient $\beta : K \to Q$ has strictly larger slope than a_{min}, because otherwise $E/\ker(\beta)$, since it is an extension of F_{max} by Q, would have smaller slope than a_{min}, or else would be of the same slope and larger rank than F_{max} by Lemma 4.0. We may assume by induction on the rank that there is a Harder-Narasimhan filtration of K, so we are done.

Definition: For fixed C, let $\mathrm{SStab}_C^{r,d}$ be the functor from schemes to sets defined by:

$$\mathrm{SStab}_C^{r,d}(S) = \left\{ \begin{array}{c} \text{locally free sheaves } \mathcal{E} \text{ of rank } r \text{ on } C \times S \text{ such that} \\ \text{for all closed } s \in S,\ \mathcal{E}|_s \text{ is semistable of degree } d \end{array} \right\} / \sim$$

where $\mathcal{E} \sim \mathcal{E}'$ if $\mathcal{E}|_s$ and $\mathcal{E}'|_s$ have the same associated graded for all closed $s \in S$.

Notice that from the definition it is immediate that if $\mathrm{SStab}_C^{r,d}$ is coarsely represented by a scheme X, and if there is a "universal" vector bundle on $C \times X$, then X is a fine moduli space.

The following theorem is due to Narasimhan and Seshadri.

Theorem 4VB: There is a projective coarse moduli space $\mathcal{M}^{r,d}(C)$ for the functor $\mathrm{SStab}_C^{r,d}$. Moreover, if d and r are relatively prime, then $\mathcal{M}^{r,d}(C)$ is a fine moduli space.

We begin with two key lemmas, the first solving the relevant GIT problem, and the second having to do with the boundedness of certain families of sheaves on a smooth curve. The second lemma poses difficulties when we try to generalize to higher dimension.

(Linear Algebra) Lemma 4.3: If V and W are vector spaces and M is an integer, then a point $\psi \in G(V \otimes W, M)$ (the Grassmannian parametrizing quotients $V \otimes W \to T$ of rank M) is semistable (resp. stable) with respect to the canonical linearization of $\mathrm{SL}(V)$ if and only if for every subspace $H \subset V$, we have:

(i) $H \otimes W$ is not in the kernel of ψ and

(ii) $\frac{\dim(H)}{\dim(\psi(H \otimes V))} \leq \frac{\dim(V)}{\dim(T)}$ (resp. $<$).

Proof: Let $N = \dim(V)$. The action of $\mathrm{SL}(V)$ on the Grassmannian is linearized by the induced action on the dual space of $\wedge^M(V \otimes W)$. If a basis $e_1, ..., e_N$ of V is given, then we say that the Plücker vector $e_{i_1} \otimes w_1 \wedge ... \wedge e_{i_M} \otimes w_M$ contains the (not necessarily distinct!) basis vectors $e_{i_1}, ..., e_{i_M}$. If $\lambda = \mathrm{diag}\{t^{r_1}, ..., t^{r_N}\}$ is a 1-PS of $\mathrm{SL}(V)$ with respect to this basis, then the weight of the Plücker vector above is $\sum_{j=1}^{M} r_{i_j}$. An element ψ of the

Grassmannian is λ-unstable if and only if $\wedge^M\psi$ vanishes on every Plücker vector of nonpositive weight.

Suppose that there is a nontrivial $H \subset V$ such that $H \otimes W$ is in the kernel of ψ. Then let $e_1, ..., e_N$ be a basis of V such that $e_1 \in H$, and let λ be the 1-PS diag$\{t^{1-N}, t, ..., t\}$ of SL(V). Then ψ is λ-unstable, because every Plücker vector of nonpositive weight must contain e_1, hence $\wedge^M\psi$ vanishes on it.

Suppose that $H \subset V$ is of dimension n, $\dim(\psi(H \otimes W)) = m_n$, and $\frac{n}{N} > \frac{m_n}{M}$. Then let $e_1, ..., e_n$ be a basis of H, and extend it to a basis $e_1, ..., e_N$ of V. Let $\lambda = \text{diag}\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$. If $\wedge^M\psi$ does not vanish on some Plücker vector, then that vector must contain at most m_n of the $e_1, ..., e_n$, by assumption, so its weight must be at least $m_n(n-N) + (M-m_n)n > 0$, and so $\wedge^M\psi$ is λ-unstable, and we see that if ψ is semistable, then (i) and (ii) must hold.

Conversely, suppose that ψ is λ-unstable for some 1-PS λ of the form diag$\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$. Let $H \subset V$ be the hyperplane spanned by $e_1, ..., e_n$. Since $\wedge^M\psi$ must vanish on any Plücker vector of nonpositive weight, it follows that either $\psi(V \otimes W) \neq T$, or else the dimension $m_n := \dim(\psi(H \otimes W))$ must satisfy $m_n(N-n) < (M-m_n)n$, so (i) or (ii) must fail.

Finally, suppose λ is a general 1-PS, diagonalized as $\lambda = \text{diag}\{t^{r_1}, ..., t^{r_N}\}$ with respect to a basis $e_1, ..., e_N$ of V, and suppose that ψ is λ- unstable. Let H_n be the span of $e_1, ..., e_n$, and let $m_n = \dim(\psi(H_n \otimes W))$. Assuming (as we may) that $\psi(V \otimes W) = T$, we have:

$$(*)\ L(r_1, ..., r_N) := r_1m_1 + r_2(m_2 - m_1) + ... + r_N(M - m_{N-1}) > 0$$

because the left side is the minimal weight of a Plücker vector on which $\wedge^M\psi$ is nonzero, and ψ is λ-unstable.

But $L(r_1, ..., r_N)$ is linear in the r_i, which are nondecreasing and sum to zero. So $(*)$ holds if and only if $L(n-N, ..., n-N, n, ..., n) > 0$ for some n, and it follows that ψ is also λ-unstable for this choice of weights, which completes the proof of the lemma for semistability, and stability is checked in the same way.

(Boundedness) Lemma 4.4: Let $p \in C$, and $\mathcal{O}_C(1) := \mathcal{O}_C(p)$. Then for each degree d and rank r (or equivalently, for each Hilbert polynomial $P(n) = rn + d - r(g-1)$), there is an integer N such that $n \geq N$ implies:

(a) If E is a semistable bundle of rank r and degree d, then $\mathrm{H}^1(C, E(n))$ vanishes, and $E(n)$ is generated by global sections.

(b) If E is a semistable bundle of rank r and degree d and $F \subset E$ is any subbundle, then we have:

$$\frac{\mathrm{h}^0(C, F(n))}{\mathrm{rank}(F)} \leq \frac{\mathrm{h}^0(C, E(n))}{\mathrm{rank}(E)}$$

Moreover, equality implies that:

$$\frac{\chi(C, F(m))}{\mathrm{rank}(F)} = \frac{\chi(C, E(m))}{\mathrm{rank}(E)}$$

for all m.

(c) If $\mathcal{E}$ is a coherent sheaf on C of Hilbert polynomial $P(n)$ such that every vector bundle quotient $\mathcal{E} \to G$ satisfies:

$$\frac{\mathrm{h}^0(C, G(n))}{\mathrm{rank}(G)} \geq \frac{P(n)}{r},$$

then $\mathcal{E}$ is a semistable vector bundle.

Proof: The key point for curves is the following. If F is a semistable bundle of rank r' and $h^0(C, F) \neq \chi(C, F)$, then $h^0(C, F) \leq r'g$. (Reference?)

Choose $N > 2g - 1 - \frac{d}{r}$. We already saw that (a) is satisfied in Example (iv). Moreover, for $n \geq N$, any semistable bundle F of rank $r' \leq r$ and slope $\mu \leq \frac{d}{r}$ must satisfy $h^0(C, F(n)) = \chi(C, F(n)) \leq \frac{r'}{r}P(n)$, or else $h^0(C, F(n)) \leq r'g < \frac{r'}{r}P(n)$.

If $F \subset E$ and E is semistable, then every subquotient in the Harder-Narasimhan filtration of F has slope at most $\frac{d}{r}$, so for $n > N$, each subquotient F_i satisfies

$$\frac{\mathrm{h}^0(C, F_i(n))}{\mathrm{rank}(F_i)} \leq \frac{P(n)}{r}$$

and by a repeated application of Lemma 4.0, we have the same inequality for F, which is the first part of (b). If equality holds, then it must hold for every subquotient F_i, and we conclude that every F_i has slope exactly $\frac{d}{r}$, so $\mu(F) = \mu(E)$ and $\frac{\chi(C,F(m))}{\mathrm{rank}(F)} = \frac{P(m)}{\mathrm{rank}(E)}$ for all m.

Finally, suppose $\mathcal{E}$ is the sheaf in (c). Let $\mathcal{T} \subset \mathcal{E}$ be the torsion subsheaf, and let G be the last (semistable) quotient in the Harder-Narasimhan filtration of $\mathcal{E}/\mathcal{T}$. Since $\mu(G) \leq \mu(\mathcal{E}/\mathcal{T}) \leq \frac{d}{r}$, it follows that:

$$\frac{\mathrm{h}^0(C, G(n))}{\mathrm{rank}(G)} \leq \frac{P(n)}{r}$$

with equality if and only if $\mathrm{h}^0(C, G(n)) = \chi(C, G(n))$, $\mathcal{E}/\mathcal{T} = G$ and $\mathcal{T} = 0$, i.e. $\mathcal{E} = G$(!)

We are ready to prove Theorem 4VB now.

Proof of Theorem 4VB: For fixed r and d, let N be chosen as in the boundedness Lemma 4.4. As in that lemma, let $P(n)$ be the Hilbert polynomial of a vector bundle of rank r and degree d. For $n > N$, consider the Quot scheme: $\mathrm{Quot}_P(V \otimes \mathcal{O}_C(-n)/C)$ (to be abbreviated $\mathrm{Quot}_{P,n}(V)$) parametrizing quotients

$$V \otimes \mathcal{O}_C(-n) \to \mathcal{E}$$

where V is a vector space of dimension $P(n)$. As we have already remarked, every semistable bundle of rank r and degree d will appear as such a quotient.

Recall that we proved in the fall that there is an M such that for all $m \geq M$, the maps $V \otimes \mathcal{O}_C(m-n) \to \mathcal{E}(m)$ are all surjective on global sections, and if we let $W = \mathrm{H}^0(C, \mathcal{O}_C(m-n))$ then we get an embedding:

$$\iota_{m,n} : \mathrm{Quot}_{P,n}(V) \hookrightarrow G(V \otimes W, P(m)).$$

We therefore choose to linearize the $\mathrm{SL}(V)$ action on $\mathrm{Quot}_{P,n}(V)$ via the canonical linearization on the Grassmannian. (Note that the linearizing line bundle is isomorphic to $\wedge^{P(m)}\pi_*\mathcal{U}(m)$, where π is the projection to the Quot scheme from its product with C, and $\mathcal{U}$ is the universal quotient sheaf.) If $x \in \mathrm{Quot}_{P,n}(V)$, let $\mathcal{E}_x$ stand for the corresponding quotient sheaf, let $\psi_x : V \otimes W \to \mathrm{H}^0(C, \mathcal{E}_x(m))$ be the induced map, and let X^U, X^{SS} and X^S

be the loci of unstable, semistable and stable points, respectively, of the Quot scheme.

Step 1: After possibly increasing M independently of x, if

(a) $\mathcal{E}_x$ is semistable and

(b) $V \to \mathrm{H}^0(C, \mathcal{E}_x)$ is an isomorphism,

then $x \in X^{SS}$.

Proof of Step 1: If $x \in X^U$, then by Lemma 4.3, there is a nonzero $H \subset V$ so that either $\psi_x(H \otimes W) = 0$, or else

$$(*) \quad \frac{\dim(H)}{\dim(\psi_x(H \otimes W))} > \frac{P(n)}{P(m)}$$

If the first holds, then in fact $H \subset V$ is in the kernel of the map $V \to \mathrm{H}^0(C, \mathcal{E}_x)$, because recall that $W = \mathrm{H}^0(C, \mathcal{O}_C(m-n))$, and ψ_x is just multiplication of sections. So (b) fails.

A point x of the Quot scheme corresponds to a quotient $V \otimes \mathcal{O}_C(-n) \to \mathcal{E}_x$. For each $H \subset V$, let $\mathcal{F}_{x,H} \subset \mathcal{E}_x$ be the subsheaf generated by $H \otimes \mathcal{O}_C(-n)$, and let $\mathcal{K}_{x,H}$ be the kernel:

$$0 \to \mathcal{K}_{x,H} \to H \otimes \mathcal{O}_C(-n) \to \mathcal{F}_{x,H} \to 0$$

Since the set of $\mathcal{F}_{x,H}$ and $\mathcal{K}_{x,H}$ vary in a bounded family (indexed by a product of the Quot scheme and Grassmannians), we can therefore choose M so that $m \geq M$ implies that $\mathrm{H}^1(C, \mathcal{K}_{x,H}(m)) = 0$ and $\mathrm{H}^1(C, \mathcal{F}_{x,H}(m)) = 0$, simultaneously for all $H \subset V$ and all x in the Quot scheme. So $\psi(H \otimes W) = \mathrm{H}^0(C, \mathcal{F}_{x,H}(m))$ is of dimension $\chi(C, \mathcal{F}_{x,H}(m))$.

Thus if $(*)$ holds, then:

$$\frac{\dim(\mathrm{H}^0(C, \mathcal{F}_{x,H}(n)))}{\chi(C, \mathcal{F}_{x,H}(m))} > \frac{P(n)}{P(m)}.$$

On the other hand, if $\mathcal{E}_x$ were semistable, then Lemma 4.4 (a) and (b) would imply that:

$$(**) \quad \frac{\dim(\mathrm{H}^0(C, \mathcal{F}_{x,H}(n))}{\mathrm{rank}(\mathcal{F}_{x,H})} \leq \frac{\dim(\mathrm{H}^0(C, E)(n))}{\mathrm{rank}(E)} = \frac{P(n)}{\mathrm{rank}(E)}$$

But $\chi(C, \mathcal{F}_{x,H}(m)) = r'm + d' - r'(g-1)$, where $r' = \text{rank}(\mathcal{F}_{x,H})$ and $d' = \deg(\mathcal{F}_{x,H})$. Similarly, $P(m) = rm + d - r(g-1)$. Then since there are only finitely many possible degrees d' (the $\mathcal{F}_{x,H}$ vary in a bounded family!), it follows that there is an M independent of x and H so that for $m > M$, a strict inequality in $(**)$ would contradict $(*)$. On the other hand, by the second part of Lemma 4.4 (b), equality in $(**)$ would also violate $(*)$, which completes the proof.

Step 2: After possibly increasing M again, if $x \in X^{SS}$ then:

(a) The map $V \to \mathrm{H}^0(C, \mathcal{E}_x(n))$ is an isomorphism and

(b) The quotient $\mathcal{E}_x$ is a semistable vector bundle.

Proof of Step 2: By Lemma 4.3, if $x \in X^{SS}$, then $V \to \mathrm{H}^0(C, \mathcal{E}_x)$ must be injective, because any kernel would yield a kernel of ψ_x after tensoring by W. Moreover, for all $H \subset V$, we must have:

$$(*) \quad \frac{\dim(H)}{\dim(\psi_x(H \otimes W))} \leq \frac{P(n)}{P(m)}$$

Suppose $\mathcal{E}_x$ were not a bundle or not semistable. Then by Lemma 4.4(c), we could find a quotient bundle $\mathcal{E}_x \to G$ so that $\frac{\mathrm{h}^0(C,G(n))}{\text{rank}(G)} < \frac{P(n)}{r}$. Then let H be the kernel of the map $V \to \mathrm{H}^0(C, G(n))$ for such a quotient, and let $\mathcal{F}_{x,H}$ be the image of H in $\mathcal{E}_x$. If $\mathcal{F}_{x,H}$ is torsion, then there is a universal bound on its length, say by K, and we can choose M so that $\frac{P(n)}{P(m)} < \frac{1}{K}$ for $m > M$, violating $(*)$.

Otherwise, by the arithmetic of Lemma 4.0, we have:

$$(**) \quad \frac{\dim(H)}{\text{rank}(\mathcal{F}_{x,H})} > \frac{P(n)}{r}$$

where the rank of $\mathcal{F}_{x,H}$ is the generic rank, or if you prefer, the coefficient of m in the Hilbert polynomial $\chi(C, \mathcal{F}_{x,H}(m))$. Since $\chi(C, \mathcal{F}_{x,H}(m)) = \dim(\psi_x(H \otimes W))$ for $m > M$ as before, then just as in Step 1, we get a contradiction to $(*)$, perhaps after boosting M again, from the fact that there is a uniform upper bound on the constant terms of the Hilbert polynomials of the $\mathcal{F}_{x,H}$. So $\mathcal{E}_x$ is semistable.

Finally, since $\mathcal{E}_x$ is semistable, the map $V \to \mathrm{H}^0(C, \mathcal{E}_x(n))$, which we already saw was injective, must be an isomorphism by Lemma 4.3(a).

Step 3: (a) $x \in X^S \iff x \in X^{SS}$ and $\mathcal{E}_x$ is stable.

(b) For any $x \in X^{SS}$, the unique closed orbit $O(x') \subset \overline{O(x)}$ corresponds to a quotient such that $\mathcal{E}_{x'}$ is isomorphic to the associated graded of $\mathcal{E}_x$.

Proof of Step 3: (a) follows the same argument as Steps 1 and 2, replacing the inequalities $(**)$ of Step 1 and $(*)$ of Step 2 by strict inequalities.

If $x \in X^{SS} - X^S$, then let $F \subset \mathcal{E}_x$ be a proper subbundle of the same slope, and let $H \subset V$ be the kernel of the map $V \to \mathrm{H}^0(C, G(n))$, where $G = \mathcal{E}_x/F$. Consider the induced extension:

$$(\dagger): \ 0 \to F \to \mathcal{E}_x \to G \to 0$$

of vector bundles of the same slope.

If we take $e_1, ..., e_n$ spanning H, extend to a basis of V, and consider the 1-PS subgroup $\lambda = \mathrm{diag}\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$, then λ acts on the extension class of $\dagger$ in $\mathrm{H}^1(C, G^* \otimes F)$ by multiplication by t^N, taking it to the split extension in the limit as $t \to 0$. We can repeat the process until we get to the associated graded of $\mathcal{E}_x$. Since the associated graded is uniquely determined by Schur's Lemma, and there must be *some* closed orbit in the closure of the orbit of $\mathcal{E}_x$, this must be the one!

Step 4: The GIT quotient $\mathcal{M}^{r,d}(C)$ of X^{SS} by $\mathrm{SL}(V)$ is a coarse moduli space for the functor $\mathrm{SStab}_C^{r,d}$.

We saw in Step 3 that the closed points of $\mathcal{M}^{r,d}(C)$ correspond to equivalence classes of semistable bundles, hence to the points of $\mathrm{SStab}_C^{r,d}(\mathrm{Spec}(\mathbf{C}))$. Suppose that $\mathcal{E}_S$ is a family of semistable bundles on $C \times S$ of rank r and relative degree d. Then we have the (surjective) evaluation map $\pi_S^*(\pi_S)_*\mathcal{E}_S(n) \to \mathcal{E}_S(n)$, which we may trivialize in neighborhoods U_s of each closed point $s \in S$ to get quotients $V \otimes \mathcal{O}_{C \times U_s}(-n) \to \mathcal{E}_{U_s}$. In other words, we get a map from each U_s to X^{SS}, and via the quotient, to $\mathcal{M}^{r,d}(C)$. Since two different trivializations differ fiberwise (after scaling) by an element of $\mathrm{SL}(V)$, the local maps glue to $S \to \mathcal{M}^{r,d}(C)$.

If there were another projective scheme Y with this property, then the universal quotient $V \otimes \mathcal{O}_{C \times X^{SS}}(-n) \to \mathcal{U}$ induces a map $X^{SS} \to Y$ which is constant on fibers, hence a uniquely determined map $\mathcal{M}^{r,d}(C) \to Y$, since $\mathcal{M}^{r,d}(C)$ is a categorical quotient, by Theorem 2P.

Step 5: $\mathcal{M}^{r,d}(C)$ is a fine moduli space if d and r are relatively prime.

Every semistable bundle is stable if r and d are relatively prime. Recall that by Schur's Lemma, if E is stable, then $\mathrm{Aut}(E) = \mathbf{C}^*$. If $g \in \mathrm{SL}(V)$ stabilizes a point $x \in X^{SS}$, then this implies g acts on $\mathcal{E}_x$ as a $P(n)$th root of unity times the diagonal, since it acts by multiplication by the same constant on every section of $\mathcal{E}_x(n)$. Since d and r are relatively prime by assumption, there are integers a and b so that $1 + ar = bP(n)$. (Recall that $P(n) = nr + d - r(g-1)$.) Thus, the action of g on $\mathcal{E}_x \otimes (\wedge^r \mathcal{E}_x)^{\otimes a}$ is trivial. An obvious (!?!) extension of the descent lemma of Kempf (Theorem 3D) implies that the twist $\mathcal{U} \otimes (\wedge^r \mathcal{U})^{\otimes a}$ of the universal subbundle on $C \times X^{SS}$ descends to a bundle on $C \times \mathcal{M}^{r,d}(C)$. On the other hand, the bundle $\wedge^r \mathcal{U}$ on $C \times X^{SS}$ determines a map $X^{SS} \to \mathrm{Pic}^d$, which must factor through the categorical quotient $\mathcal{M}^{r,d}(C)$. So by the universal property of Pic, there is a line bundle $\mathcal{L}$ on $\mathrm{Pic}^d(C)$ with the property that the pullback of $\mathcal{L}$ to $C \times X^{SS}$ coincides with $\wedge^r \mathcal{E}_x$ on each $C \times \{x\}$. (Reference?) But if we let $\det : \mathcal{M}^{r,d}(C) \to \mathrm{Pic}^d(C)$ be the induced map, then this implies that tensoring the descended vector bundle with $\det^*(\mathcal{L}^*)^{\otimes a}$ yields a universal bundle.

Proposition 4.5: There is a scheme structure on the subset

$$\Theta := \{E \mid \mathrm{H}^0(C, E) \neq 0\} \subset \mathcal{M}^{r,r(g-1)}(C)$$

making it an ample Cartier divisor.

Proof: Let n be chosen as in Lemma 4.4, let $X^{SS} \subset \mathrm{Quot}_P(V \otimes \mathcal{O}_C(-D)/C)$ be the semistable locus, where $P(m) = mr$, and $D = \sum_{i=1}^n p_i$ is a divisor on C consisting of distinct points. If $\mathcal{U}$ is the universal quotient on $C \times X^{SS}$, then pushing down the exact sequence:

$$0 \to \mathcal{U} \to \mathcal{U}(D) \to \oplus_{i=1}^n \mathcal{U}(D)_{p_i} \to 0$$

yields the sequence:

$$0 \to \pi_{X_*^{SS}} \mathcal{U} \to \pi_{X_*^{SS}} \mathcal{U}(D) \xrightarrow{f} \oplus_{i=1}^n \mathcal{U}(D)_{p_i} \to R^1 \pi_{X_*^{SS}} \mathcal{U} \to 0$$

where the middle two sheaves are both locally free of rank $N = rn$. Moreover, since there exist semistable bundles E of degree $r(g-1)$ with $\mathrm{H}^1(C, E) = 0$, (e.g. $E = \oplus^r L$ where $\mathrm{H}^1(C, L) = 0$), the first sheaf vanishes! Finally, the map f is G-invariant, so f descends, and $\wedge^N(f)$, a (nonzero) section of the line bundle $\mathcal{L} := Hom(\wedge^N \pi_{X_*^{ss}} \mathcal{U}(D), \otimes_{i=1}^n \wedge^r \mathcal{U}(D)_{p_i})$ descends to a section s which vanishes precisely on Θ. If $m > M$ is fixed, then $\mathcal{O}_X(1) := \wedge^{mr} \pi_* \mathcal{U}(m)$ is the linearization used in Theorem 4VB to define X^{SS}. In particular, some power of $\mathcal{O}(1)$ descends to an ample line bundle on $\mathcal{M}^{r,d}(C)$. We claim that there are integers a and b such that $\mathcal{L}^a$ and $\mathcal{O}(b)$ differ by the pullback of a line bundle from $\mathrm{Pic}^d(C)$. This implies that Θ is ample.

But $\wedge^N \pi_{X_*^{ss}} \mathcal{U}(D)$ is trivial, naturally isomorphic to $\wedge^N V \otimes \mathcal{O}$, and the difference between $\wedge^{cr} \pi_{X_*^{ss}} \mathcal{U}(c)$ and $\wedge^{(c+1)r} \pi_{X_*^{ss}} \mathcal{U}(c+1)$ is a translate of the bundle $\wedge^r \mathcal{U}_p$ by the pullback of a line bundle from $\mathrm{Pic}^d(C)$ ($p \in C$ is an arbitrary point). The result is therefore immediate, since up to translation, $\mathcal{L}$ and $\mathcal{O}(1)$ are powers of the same line bundle.