Course Notes for Math 780-1 (Geometric Invariant Theory)

1. Hilbert's Fourteenth Problem. Throughout the course, G will denote a linear group over $\mathbf{C}$, that is, a closed (hence affine) subgroup of $\mathrm{GL}(N, \mathbf{C})$. (Indeed, we will be taking $G = \mathrm{SL}(N, \mathbf{C})$ in virtually all the examples!)

If G acts algebraically on a vector space V of dimension n over $\mathbf{C}$ (that is, if the induced $\rho : G \to \mathrm{GL}(V)$ is a morphism), then G acts on the polynomial ring $\mathrm{Sym}(V^*) \cong \mathbf{C}[x_1, ..., x_n]$, and we will denote by $\mathbf{C}[x_1, ..., x_n]^G$ the ring of polynomials left invariant under the G-action.

More generally, if G acts on a k-algebra R by k-algebra automorphisms, then we denote by R^G the subring of invariant elements, and make the following definition:

Definition: The action of G on R is rational if every element of R is contained in a finite-dimensional subspace which is invariant under G, and on which G acts algebraically.

In particular, the induced action of G on $\mathbf{C}[x_1, .., x_n]$ above is rational. But more generally, if $X = \mathrm{Spec}(R)$ is any affine variety, and $\sigma^* : R \to S \otimes R$ is the dual action to an action $\sigma : G \times X \to X$, then for any $r \in R$, write $\sigma^*(r) = \sum s_i \otimes r_i$. Then the vector space spanned by the r_i is finite dimensional, and contains the invariant vector space spanned by Gr. So the action is rational.

The starting point for GIT is the finite-generatedness of rings of invariant elements, so it seems only fitting to introduce:

Hilbert's Fourteenth Problem: If G acts rationally on a finitely generated k-algebra R, then is the subring R^G also finitely generated?

Notes: Hilbert had already proved this for $G = \mathrm{SL}(n, \mathbf{C})$, embedded in $\mathrm{GL}(N, \mathbf{C})$ by a symmetric power of the standard representation.

Actually, it seems Hilbert thought this question had already been answered in the affirmative, so he really proposed a more general question!

Nagata's Answer: The answer is no, as stated. (See, e.g., Dieudonné and Carrell for Nagata's counterexample.) More assumptions are needed on G.

Definition: G is *linearly* reductive if for every finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ and every subspace $W \subset V$ invariant under the action of

G, there is a (complementary) invariant subspace W' such that $V = W \oplus W'$. (That is, every algebraic action of G is completely reducible.)

The following theorems date back to Weyl.

Theorem 1A: If G is linearly reductive, acting rationally on a finitely generated k-algebra R, then R^G is finitely generated.

The main tool in the proof of Theorem 1A is the existence and properties of the **Reynolds operator**:

Lemma 1.1: If G is linearly reductive and acts rationally on a k-vector space V (i.e. every $v \in V$ is contained in a finite-dimensional invariant subspace on which G acts algebraically), let V^G be the subspace of invariant vectors.

Then there is a uniquely defined linear operator $E : V \to V$ projecting V onto V^G. This is called the Reynolds operator.

Moreover, if $u : V \to V'$ is a G-linear map of vector spaces on which G acts rationally, then the Reynolds operators for V and V' commute with u.

Proof: If $v \in V$ is not invariant, let W be a finite-dimensional invariant subspace containing v, and decompose $W = W^G \oplus W_G$ by the linear reductivity of G. Then $v \in W_G$, so W_G is nonempty, invariant and $W_G \cap V^G = \emptyset$. So Zorn's lemma applies to the set of invariant subspaces $T \subset V$ with $T \cap V^G = \emptyset$. Let V_G be a maximal such.

Some mucking around (exercise!) shows that V_G is uniquely determined by this property and that $V^G \oplus V_G = V$. Thus, the Reynolds operator E is uniquely defined by the property that has V_G as its kernel and fixes V^G.

Let E' be the Reynolds operator for V' and E be the Reynolds operator for V. In order to show that $E' \circ u = u \circ E$, it suffices to show that $u(V^G) \subset (V')^G$ and $u(V_G) \subset (V')_G$. The first inclusion is obvious. For the second, suppose that $v \in V_G$, and let W be a finite-dimensional invariant subspace of V_G containing v. Then $W \cap \ker(u)$ is invariant, so by linear reductivity we can decompose $W = (W \cap \ker(u)) \oplus W'$ where W' is invariant. But now u maps W' isomorphically onto $u(W') = u(W)$, hence $u(W)$ is invariant and $u(W) \cap (V')^G = 0$, so $u(W) \subset (V')_G$, so $u(v) \in (V')_G$, as desired.

Corollary 1.2: If the G-linear map in the Lemma is surjective, then the induced map $u^G : V^G \to (V')^G$ is also surjective.

Proof: By the lemma,

$$(V')^G = E'(V') = E'(u(V)) = u(E(V)) = u(V^G)$$

Corollary 1.3 (The Reynolds Identity): If G is linearly reductive and acts rationally on the k-algebra R (hence on the k-vector space R), then for all $x \in R^G$ and $y \in R$, we have:

$$E(xy) = xE(y)$$

Proof: The map $y \mapsto xy$ is a G-linear automorphism of the vector space R, so the Lemma applies.

Corollary 1.4: If G is linearly reductive, acting rationally on the k-algebra R, and if (I_i) is a family of invariant ideals in R, then

$$(\sum_i I_i) \cap R^G = \sum_i (I_i \cap R^G)$$

Proof: Thinking of I_i as a subspace of R on which G acts, it follows from the Lemma that the restriction of the Reynolds operator from R coincides with the Reynolds operator on I_i. In particular, $E(f_i) \in I_i \cap R^G$ for all $f_i \in I_i$. So if $f \in (\sum I_i) \cap R^G$, then $f = \sum f_i$ is a finite sum, with $f_i \in I_i$, and

$$f = E(f) = \sum_i E(f_i) \in \sum_i (I_i \cap R^G).$$

The other inclusion is obvious.

Corollary 1.4 will be used in the next section.

Proof of Theorem 1A: Let $f_1, ..., f_r$ be generators of R, and let V be a finite-dimensional invariant subspace containing the generators (which is guaranteed to exist since the action is rational). Then under the induced action of G on $S = \mathrm{Sym}(V)$, the surjective map $u : S \to R$ commutes with the action of G. By Corollary 1.2, the induced map $u^G : S^G \to R^G$ is also surjective. Thus it suffices to prove the theorem for the action of G the symmetric algebra S.

Since this action of G on S preserves degrees, the ring of invariants S^G is graded. Say $S^G = \sum_{d \geq 0} S^G_d$, and let I be the ideal in S generated by the positive-degree invariants $\sum_{d>0} S^G_d$. Then I is finitely generated over S, and the generators $M_1, ..., M_m$ may of course be taken to be homogeneous and invariant.

We claim that $1, M_1, ..., M_m$ generate S^G as a k-algebra. Indeed, by induction (the case $d = 0$ being trivial), we may assume that $1, M_1, ..., M_m$ generate S^G in degree less than d. If P is homogeneous of degree d, we can write $P = \sum Q_i M_i$, for $Q_i \in S$. and by Corollary 1.3, we have

$$P = R(P) = \sum R(Q_i)M_i$$

Since the degrees of the $R(Q_i)$ are all smaller than d, they are in the algebra generated by $1, M_1, ..., M_m$, and we are done.

Theorem 1B: $\mathrm{SL}(N, \mathbf{C})$ is linearly reductive.

Proof: (This is known as Weyl's unitary trick.) First observe that the special unitary group $\mathrm{SU}(N) \subset \mathrm{SL}(N, \mathbf{C})$ is linearly reductive. This is basically because $\mathrm{SU}(N)$ is compact. Indeed, if $\mathrm{SU}(N)$ acts on a finite-dimensional vector space V, then choose a positive definite Hermitian inner product h on V. Since $\mathrm{SU}(N)$ is compact, we can average h over it to produce an $\mathrm{SU}(N)$-invariant positive definite Hermitian inner product H on V. But now if $W \subset V$ is an invariant subspace, then the orthogonal complement of W with respect to H is also invariant, and so $\mathrm{SU}(N)$ is linearly reductive.

Next, observe that $\mathrm{SU}(N)$ is Zariski dense in $\mathrm{SL}(N, \mathbf{C})$. Indeed, the Zariski tangent space to $\mathrm{SU}(N)$ consists of the traceless matrices A satisfying $A = -\overline{A^t}$. The complex span of these is all traceless matrices, so the tangent space to the Zariski closure of $\mathrm{SU}(N)$ at the origin fills the tangent space to $\mathrm{SL}(N, \mathbf{C})$, and so they coincide.

Suppose $W \subset V$ is $\mathrm{SU}(N)$-invariant. Then the stabilizer of W in $\mathrm{SL}(N, \mathbf{C})$ is Zariski closed, and contains $\mathrm{SU}(N)$, so it must be all of $\mathrm{SL}(N)$. Thus the invariant subspaces are the same for the two groups, so the linear reductivity of $\mathrm{SL}(N, \mathbf{C})$ follows from the linear reductivity of $\mathrm{SU}(N)$.

Exercise: Show that the torus $\mathbf{G}_m(= \ \mathrm{GL}(1))$ is linearly reductive, indeed show that any representation space V may be decomposed as a sum of one-dimensional invariant subspaces.

In particular, Hilbert's fourteenth problem is true for any representation of $\mathrm{SL}(N, \mathbf{C})$ by Theorems 1 and 2. Indeed, Weyl's unitary trick can be souped up to show that all connected semi-simple groups over $\mathbf{C}$ are linearly reductive, which gives a very satisfactory result over the complex numbers. However, over fields of characteristic $p > 0$, even the groups $\mathrm{SL}(N, k)$ are not linearly reductive. To appease the characteristic p-lovers in the audience, and also to clear up possible misconceptions based upon the confusing notation in the literature, we remark on some improvements to Theorems 1 and 2 which extend the results to positive characteristic:

Definition: G is *geometrically* reductive if for every rational representation $\rho : G \to \mathrm{GL}(V)$, and every invariant nonzero vector $v \in V$, there is an invariant homogeneous polynomial $P \in \mathrm{Sym}(V^*)$ of positive degree such that $P(v) \neq 0$.

Easy exercise: Show that linearly reductive implies geometrically reductive, where the polynomial P may be chosen to be linear. (Hence the terminology!)

Theorem (Nagata): If G is geometrically reductive, acting rationally on a finitely generated k-algebra R, then R^G is finitely generated.

Finally, there is the groupy definition of reductivity (taken from Borel), which has the advantage of being easy to check (and true!) for such groups as $\mathrm{SL}(N, k)$ and $\mathrm{GL}(N, k)$ in positive characteristic.

Definition: G is reductive if its radical (that is, the unique maximal connected normal solvable subgroup of G) is a torus.

Theorem (Haboush): G is geometrically reductive if and only if it is reductive.

This was known as Mumford's conjecture before (and even after!) Haboush proved it.