Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

0.4. Cohomology. If X is a topological space, then the global section functor:

$$\Gamma : Ab(X) \to Ab;\ \mathcal{A} \mapsto \Gamma(X, \mathcal{A}) = \mathcal{A}(X)$$

from sheaves of abelian groups on X to abelian groups is left (but not right) exact. The lack of right exactness is addressed with right derived functors:

$$\mathrm{H^i} : Ab(\mathrm{X}) \to Ab \text{ with } \mathrm{H}^0(\mathrm{X}, \mathcal{A}) = \Gamma(\mathrm{X}, \mathcal{A}) = \mathcal{A}(\mathrm{X})$$

and connecting homomorphisms δ attached to short exact sequences of sheaves:

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

that yield long exact sequences of abelian "cohomology" groups:

$$0 \to H^0(X, \mathcal{A}') \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{A}'') \stackrel{\delta}{\to} H^1(X, \mathcal{A}') \to H^1(X, \mathcal{A}) \to \cdots$$

A sheaf $\mathcal{C} \in \mathcal{A}b(X)$ is **acyclic** if $\mathrm{H}^i(X, \mathcal{C}) = 0$ for all $i > 0$, and a resolution:

$$0 \to \mathcal{A} \to \mathcal{C}^0 \stackrel{d_0}{\to} \mathcal{C}^1 \stackrel{d_1}{\to} \mathcal{C}^2 \to \cdots$$

of $\mathcal{A} = \ker(d_0)$ by acyclic sheaves is used to compute the cohomology groups via:

$$\mathrm{H}^i(X, \mathcal{A}) = \ker(\Gamma(d^i))/\mathrm{im}(\Gamma(d^{i-1}))$$

The Cech resolution is one example. If $U_i \subset X$ are the sets of an open cover and each restriction $\mathcal{A}|_{U_I}$ is acyclic as a sheaf of abelian groups on $U_I = U_{i_1} \cap \cdots \cap U_{i_n}$ for all multi-indices I, then there is a Cech resolution formed from the pushforwards:

$$0 \to \mathcal{A} \to \oplus \iota_* \mathcal{A}|_{U_i} \to \oplus \iota_* \mathcal{A}|_{U_i \cap U_j} \to \cdots$$

where ι are the inclusion maps of the open sets. The d maps are:

$d_0(..., a_i, ...) = (..., a_i - a_j, ...)$

$d_1(...., a_{ij}, ...) = (......, a_{ij} - a_{ik} + a_{jk}, ...)$

etc.

Theorem (Grothendieck). If X is a Noetherian topological space, then:

$$\mathrm{H}^i(X, \mathcal{A}) = 0 \text{ for all } \mathcal{A} \text{ and all } i > n$$

where n is the Noetherian dimension of X.

Remark. If X is a variety of finite type over k, the Noetherian dimension of X is the transcendence degree of $k(x)$ over k, where $x \in X$ is the generic point. If X is a scheme of finite type, it is the dimension of the largest irreducible component.

We may carry this over to quasicoherent sheaves on a (separated) scheme X, say of finite type over k, in which case the derived functors are vector spaces over k and we have the following:

(i) Quasicoherent sheaves on affine schemes $\mathrm{Spec}(A)$ are acyclic, i.e.

$$\mathrm{H}^i(\mathrm{Spec}(A), \widetilde{M}) = 0 \text{ for } i > 0 \text{ and}$$

(ii) The intersection of open affine subsets of a separated scheme is also affine.

This gives acyclic Cech resolutions of a quasicoherent sheaf via affine open covers.

Remark. If X is proper and $\mathcal{F}$ is coherent, cohomology spaces are finite dimensional.

Examples. The cohomology spaces of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n_k}(d)$ satisfy:

$$\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}(d)) = k[x_0, ..., x_n]_d$$
$$\mathrm{H}^i(\mathbb{P}^n_k, \mathcal{O}(d)) = 0 \text{ for all } d \text{ and } i = 1, ..., n-1$$

and finally,

$$\mathrm{H}^n(\mathbb{P}^n_k, \mathcal{O}(-n-1-d)) = k[x_0, ..., x_n]_d^*$$

(the dual vector space) via the Serre Duality pairing with $\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(d))$.

Serre Duality I. If X is a non-singular projective variety over k and $\dim(X) = n$, then:

$$\omega_X = \wedge^n \Omega_{X/k}$$

is a dualizing line bundle, in the sense that $\mathrm{H}^n(X, \omega_X) = k$ and the cup product:

$$\mathrm{H}^i(X, E) \times \mathrm{H}^{n-i}(X, E^* \otimes \omega_X) \to \mathrm{H}^n(X, E \otimes E^* \otimes \omega_X)$$

followed by the trace map (obtained from) $tr : E \otimes E^* \to \mathcal{O}_X$:

$$\mathrm{H}^n(X, E \otimes E^* \otimes \omega_X) \to \mathrm{H}^n(X, \omega_X) = k$$

is a perfect pairing, inducing an isomorphism: $\mathrm{H}^i(X, E) \cong \mathrm{H}^{n-i}(X, E^* \otimes \omega_X)^*$.

Examples. When $X = \mathbb{P}^n_k$, then $\omega_X = \mathcal{O}_X(-n-1)$ (by the Euler sequence).

When $i : X \subset Y$ is an embedding of non-singular varieties of codimension c, then the conormal sheaf $N^*_{X/Y} = i^*\mathcal{I}_X/\mathcal{I}_X^2$ is locally free of rank c, and there is an **adjunction formula**:

$$\omega_X = i^*\omega_Y \otimes \wedge^c N_{X/Y}$$

Thus, for example,

$$\omega_X = \mathcal{O}_X(-n-1+d_1+\cdots+d_c)$$

when $X \subset \mathbb{P}^n_k$ is a complete intersection of hypersurfaces of degrees $d_1, ..., d_c$.

Serre Theorem B. A line bundle $\mathcal{L}$ on a Noetherian X is ample if and only if:

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$$

for all quasi-coherent sheaves $\mathcal{F}$ on X and all $i > 0$ and $d \geq d_{\mathcal{F}}$.

Thus, in particular, if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then:

$$\mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0 \text{ for all } i > 0 \text{ and } d \geq d_{\mathcal{F}}$$

It is natural to ask for a specific value of $d_{\mathcal{F}}$ and to ask whether $\mathcal{F} \otimes \mathcal{L}^d$ is also generated by global sections when $d \geq d_{\mathcal{F}}$ (from Serre's other theorem). A priori, this involves checking an infinite number of conditions! But there is a better way.

Definition. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is Castelnuovo-Mumford **d-regular** if:

$$\mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(d-i)) = 0 \text{ for all } i > 0$$

Theorem (Mumford) If $\mathcal{F}$ is d-regular, then:

(a) $\mathcal{F}$ is $d+1$-regular, and

(b) The multiplication map:

$$\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}(1)) \times \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(d)) \to \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$$

is surjective (from which it follows that $\mathcal{F}(d)$ is generated by global sections).

Remark. If $\mathcal{F}$ is d-regular, it follows that $d = d_{\mathcal{F}}$ suffices for the vanishing in Serre's Theorem A and B with only finitely many vanishings of cohomology to check.

Proof. Let $\mathbb{P}^n_k = \mathbb{P}(V)$. Then the Euler sequence continues to the left:

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-n-1) \to \cdots \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(V)}(-2) \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0$$

as a **Koszul complex**. Tensoring by $\mathcal{F}(d)$ gives a long exact sequence:

$$(*)_d \quad 0 \to \mathcal{F}(d-n-1) \to \cdots \to \wedge^2 V \otimes \mathcal{F}(d-2) \to V \otimes \mathcal{F}(d-1) \stackrel{m}{\to} \mathcal{F}(d) \to 0$$

of coherent sheaves, where m is the muliplication map on sections.

A **spectral sequence** now connects each of the individual cohomology spaces. This applies to any long exact sequence of coherent sheaves:

$$0 \to \mathcal{F}_{-n} \to \cdots \to \mathcal{F}_0 \to 0$$

and generalizes the long exact sequence of cohomology spaces associated to a short exact sequence of coherent sheaves. It presents itself as a series of tableaux.

The E^1 Tableau (horizontal arrows).

$$E^1_{-p,q} = \mathrm{H}^q(\mathbb{P}^n, \mathcal{F}_{-p})$$

with

$$d^1_{-p,q} : E^1_{-p,q} \to E^1_{-p+1,q}$$

from which:

The E^2 Tableau (knight's move arrows..two over and one down) is defined by:

$$E^2_{-p,q} = \ker(d^1_{-p,q})/\mathrm{im}(d^1_{-p+1,q})$$

with induced arrows

$$d^2_{-p,q} : E^2_{-p,q} \to E^2_{-p+2,q-1}$$

and then inductively:

The E^k Tableau

$$E^{k+1}_{-p,q} = \ker(d^k_{-p,q})/\mathrm{im}(d^k_{-p+k,q-k-q})$$

with induced arrows

$$d^{k+1}_{-p,q} : E^k_{-p,q} \to E^k_{-p+k+1,q-k}$$

and finally (the punchline)

The E^∞ Tableau is the Zero Tableau, i.e.

$$E^k_{p,q} = 0 \text{ for all sufficiently large values of } k$$

The proof now follows by noting that all maps in and out of the terms:

$$E^1_{0,q} = \mathrm{H}^q(\mathbb{P}^n, \mathcal{F}(d+1-q)) \text{ for the sequence } (*)_{d+1-q} \text{ are zero}$$

when $q > 0$ and the sole map in or out of the $E^1_{0,0}$ term for $(*)_{d+1}$ is:

$$d^1_{-1,0} : V \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d)) \to \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d+1)$$

which must therefore be surjective. □

Corollary. $\mathcal{F}$ is d-regular if and only if there is a resolution of $\mathcal{F}(d)$ of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-n)^{r_n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{r_1} \to \mathcal{O}^{r_0}_{\mathbb{P}^n} \to \mathcal{F}(d) \to 0$$

In particular, $\mathcal{F}(d)$ is generated by global sections (but this says much more).

Remark. Via Beilinson's resolution of the diagonal, one can compute:

$$r_i = \dim \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F} \otimes \wedge^i \Omega_{\mathbb{P}^n_k}(i))$$

There are two interesting features of this "linear" resolution of $\mathcal{F}(d)$:

(a) It determines a long exact resolution of $\mathcal{F}$ by locally free sheaves.

(b) It is the sheafification of **the** minimal free resolution of the graded module:

$$M(d)_{\geq 0} = \bigoplus \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(d))$$

by free graded modules:

$$0 \to \cdots \to S(-1)^{r_1} \to S^{r_0} \to M(d)_{\geq 0} \to 0$$

Hilbert's Syzygy Theorems.

(a) Every coherent sheaf $\mathcal{F}$ on a *nonsingular* variety X of finite type over k and dimension n has a resolution of length $k \leq n$ by locally free sheaves:

$$0 \to E_k \to \cdots \to E_0 \to \mathcal{F} \to 0$$

(b) Every finitely generated graded module $M_\bullet$ over $S = k[x_0, ..., x_n]_\bullet$ has a resolution by free graded modules (direct sums of modules $S(d)$) of length $k \leq n+1$.

Neither is uniquely determined by the coherent sheaf $\mathcal{F} = \widetilde{M}$, though in (b), there is a **minimal** free resolution attached to each module M associated to $\mathcal{F}$.

Example. Let $\mathcal{I}_Z$ be the ideal sheaf of a length three subscheme $Z \subset \mathbb{P}^2$. Then the minimal free resolution of the graded ideal $I_Z \subset S$ is:

(i) $0 \to S(-4) \to S(-3) \oplus S(-1) \to I_Z \to 0$ if Z is collinear ($\mathcal{I}_Z$ is not 2-regular).

(ii) $0 \to S(-3)^2 \to S(-2)^3 \to I_Z \to 0$ if Z is not collinear ($\mathcal{I}_Z$ is 2-regular).

but the resolution of the *truncated* ideal $(I_Z)_{\geq 3}$ in both cases has the form:

$$0 \to S(-5)^3 \to S(-4)^9 \to S(-3)^7 \to (I_Z)_{\geq 3} \to\to 0$$

reflecting the fact that all ideal sheaves are 3-regular.

The Higher $\mathcal{T}or$ Functors are the left derived functors for the right exact functor:

$$\otimes\mathcal{F} : Coh(X) \to Coh(X)$$

with respect to any given coherent sheaf $\mathcal{F}$.

In this setting, a resolution of $\mathcal{F}$ by *locally free sheaves* is acyclic, and:

$$\mathcal{T}or_i(\mathcal{G} \otimes \mathcal{F}) = \ker(E_i \otimes \mathcal{G} \to E_{i-1} \otimes \mathcal{G})/\mathrm{im}(E_{i+1} \otimes \mathcal{G} \to E_i \otimes \mathcal{G})$$

for any locally free resolution

$$\cdots \to E_1 \to E_0 \to \mathcal{F} \to 0$$

(which may be infinite if X is singular).

Closely related are the higher $\mathcal{E}xt$ functors for the left exact $\mathcal{H}om$ functor:

$$\mathcal{H}om(\mathcal{F}, \bullet) : \mathcal{C}oh(X) \to \mathcal{C}oh(X)$$

that are computed using the same resolution of $\mathcal{F}$ by locally free sheaves since locally free sheaves are acyclic for this functor. Notice that this resolution would seem to be pointing in the wrong direction (compare it with the acyclic resolutions of the left exact Γ functor). But the point here is that $\mathcal{H}om(\mathcal{F}, \bullet)$ is **contravariant** in $\mathcal{F}$, so arrows are reversed.

On the other hand, the **Global Ext** functors are higher derived functors for:

$$\mathrm{Hom}(\mathcal{F}, \bullet) : \mathcal{C}oh(X) \to \mathcal{V}ec(k)$$

which is the (left-exact) composition of left-exact functors:

$$\mathrm{Hom}(\mathcal{F}, \bullet) = \Gamma \circ \mathcal{H}om(\mathcal{F}, \bullet)$$

and are the correct context for:

Serre Duality II. For any pair of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a non-singular projective scheme of dimension n, there is a perfect pairing:

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X) \to k$$

Remark. A spectral sequence relates the Ext^k functors to the functors:

$$\mathrm{H}^i \circ \mathcal{E}xt^j$$

In one case, though, the relation is clear. When F is locally free, we have:

$$\mathrm{Ext}^i(F, \mathcal{G}) = \mathrm{H}^i(\mathcal{H}om(F, \mathcal{G})) = \mathrm{H}^i(X, F^* \otimes \mathcal{G})$$

since F is acyclic for the $\mathcal{H}om$ functor.

Remark. Serre duality applies in far more generality than non-singular varieties.

Finally, we have the **Higher Direct Images** for the left-exact push-forward

$$f_* : \mathcal{C}oh(X) \to \mathcal{C}oh(S) \text{ associated to a proper morphism } f : X \to S$$

In this "relative cohomology" setting, affine **morphisms** are acyclic, and one can find Cech resolutions that compute the higher right derived functors :

$$R^i f^* : \mathcal{C}oh(X) \to \mathcal{C}oh(S)$$

and as noted earlier, coherent sheaves that are flat over S play a special role, with one very important and useful theorem being the following:

Cohomology and Base Change. If $f : X \to S$ is a projective morphism of Noetherian schemes and $\mathcal{F}$ is a coherent sheaf on X that is flat over S, then the maps to the cohomology of fibers:

$$\phi^i(y) : R^i f_* \mathcal{F} \otimes k(y) \to \mathrm{H}^i(X_y, \mathcal{F}|_{X_y})$$

satisfy the following:

If $\phi^i(y)$ is surjective, then

(a) $\phi^i(y')$ is an isomorphism for all y' in a neighborhood of y, and

(b) the following are equivalent:

- (i) $\phi^{i-1}(y)$ is also surjective at y.
- (ii) $R^i f_*(\mathcal{F})$ is locally free near y.

Remark. This theorem is the crucial step in proving the characterization of flat coherent sheaves over projective morphisms in the previous section.