Abstract Algebra. Math 6320. Bertram/Utah 2022-23.
Group Representations

Let G be a group and k be a field.

Definition. A *representation* (over k) of G is a group homomorphism:

$$\rho : G \to \ \mathrm{GL}(V)$$

where V is a k-vector space and $\mathrm{GL}(V) = \ \mathrm{Aut}_k(V)$ is the group of k-automorphisms, The dimension of V is the dimension of the representation (V, ρ) (or just ρ).

In other words a representation is an *action*:

$$G \times V \to V, \text{ which we write as } (g, v) \mapsto gv := \rho(g)(v)$$

and which satisfies $g(v + w) = gv + gw$ and $(gh)v = g(hv)$. This resembles the structure of a module, and a group representation is also called a G-module.

Examples. Group characters are one-dimensional (abelian) representations.

Remarks. (a) If V has a basis $e_1, ..., e_n$, then in terms of the basis, a representation is a collection of invertible $n \times n$ matrices. When the basis is understood, $\mathrm{GL}(V)$ is called $\mathrm{GL}(n, k)$, the group of invertible $n \times n$ matrices with entries in the field k.

(b) If G is given in terms of generators g_i and relations r_j, then ρ may be specified by choosing matrices $A_i \in \ \mathrm{GL}(n, k)$ that satisfy the relations r_j. Thus, for example, a representation of the cyclic group C_d is the choice of a single $n \times n$ matrix $\rho(g) = A$ (for a generator g of C_d) satisfying $A^d = I_n$.

Example. Transpositions $g_1 = (1\ 2)$ and $g_2 = (2\ 3)$ generate S_3 with relations:

$$g_1^2 = \mathrm{id}, g_2^2 = \mathrm{id} \text{ and } (g_1 g_2)^3 = \mathrm{id}$$

Letting

$$\rho(g_1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

we check that:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^2 = I_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2$$

and

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ satisfies } \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^3 = I_2$$

so this determines a two-dimensional representation of S_3 in any field, including, amusingly, the field $\mathbb{F}_2$ (in which $-1 = 1$). For example,

$$\rho((1\ 2\ 3)) = \rho((1\ 2)(2\ 3)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ and}$$

$$\rho((1\ 3)) = \rho((1\ 2)(2\ 3)(1\ 2)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Definition. Given a representation (V, ρ) of G, a subspace $U \subset V$ is *invariant* if the image of $\rho|_U$ lies in U, in which case ρ "induces" the representation $(U, \rho|_U)$. Evidently, both $0, V \subset V$ are invariant subspaces, and if these are the only invariant subspaces, then we say (V, ρ) is an *irreducible* representation.

Example. One-dim'l characters are irreducible (there are no other subspaces!). The example above is an irreducible two-dimensional representation of S_3.

Note. The one-dimensional invariant subspaces of (V, ρ) are spanned by a shared eigenvector of the matrices $A_g = \rho(g)$ for all elements $g \in G$.

We've already seen that if G is finite, there are at most $|G|$ *characters* of G, and exactly $|G|$ of them if $k = \mathbb{C}$ and G is abelian. This begs the following:

Question. How many *irreducible* representations are there for a fixed G and k?

The answer depends upon the field k, as seen with the following example.

Example. The *rotation* representation of C_n on $\mathbb{R}^2$ generated by:

$$\rho(g) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}$$

is irreducible because this rotation (by $2\pi/n$) has no real eigenvectors. On the other hand, every automorphism of $\mathbb{C}^2$ has a complex eigenvector, and here there are two: $(1, i)$ and $(1, -i)$ with eigenvalues $e^{-2\pi i/n}$ and $e^{2\pi i/n}$, respectively. Alternatively, one could view this rotation as a single *complex* character given by $\chi(g) = e^{2\pi i/n}$.

The *category* $GRep_k$ of representations of G for a field k consists of:

- Objects (V, ρ), the k-representations of G, including the zero representation.
- Vector spaces $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_k(V, W)$ of *G-linear maps* between objects (V, ρ) and (W, τ), where a k-linear map $f : V \to W$ is G-linear if:

$$f(gv) = gf(v) \text{ for all } g \in G$$

This is an *abelian category*. The key observation is:

Proposition 1. The kernel, image and cokernel of a G-linear map $f \in \mathrm{Hom}_G(V, W)$ are also representations of G.

Proof. The kernel $K \subset V$ is an invariant subspace (hence a representation), since if $f(v) = 0$, then $f(gv) = gf(v) = g0 = 0$. Similarly, the image $I = f(V) \subset W$ is invariant, since if $w = f(v)$, then $gw = f(gv)$. Finally, the cokernel is:

$$W/I = \{w + f(V)\} \text{ and } g(w + f(V)) = gw + f(V)$$

and a well-defined action of G, with G-linear map $q : W \to W/I$. □

The rest of the properties of an abelian category follow from the corresponding properties for the category of k-vector spaces. In particular, the *direct sum* of representations (V, ρ) and (W, τ) is the direct sum of the vector spaces, together with the "diagonal" action $g(v, w) = (gv, gw)$.

But consider the following subtlety:

Example. The complex representation ρ of $G = (\mathbb{Z}, +)$ on $\mathbb{C}^2$ given by:

$$\rho(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

has one invariant subspace, the line spanned by e_1, which is a common eigenvector (with eigenvalue 1) of each $\rho(n)$. This gives a G-invariant map $\mathbb{C} \to \mathbb{C}^2$ from the trivial representation, whose cokernel is *also* the trivial representation. But the representation ρ itself is **not** trivial, and therefore not isomorphic to the direct sum of the kernel and cokernel representations. Thus, like the case of R-modules, a short exact sequence of representations (G-modules) need not have a right splitting.

On the other hand, when G is finite, we have the following:

Proposition 2. If G is a finite group and $\rho : G \to \mathrm{Aut}(V)$ is a (finite dimensional) complex representation, then every invariant subspace $U \subset V$ has an invariant complementary subspace W with $U \oplus W = V$ *as complex representations of* G.

Proof. The idea is to construct a Hermitian inner product on V that is invariant by averaging over the group, and then to take the orthogonal complement with respect to this averaged inner product. Choose a basis $e_1, ..., e_n$ of V and let:

$$\langle u, v\rangle_G = \frac{1}{|G|}\sum_{h\in G}\langle hu, hv\rangle \text{ for vectors } u, v \in V$$

where $\langle\cdot,\cdot\rangle$ is the "ordinary" Hermitian inner product: $\langle\sum x_i e_i, \sum y_i e_i\rangle = \sum x_i \overline{y}_i$.

This is rigged so that $\langle u, v\rangle_G = \langle gu, gv\rangle_G$ for all $g \in G$ and if $v \neq 0$, then

$$\langle v, v\rangle_G = \frac{1}{|G|}\sum_{g\in G}|gv|^2 > 0$$

i.e. $\langle\cdot,\cdot\rangle_G$ is positive definite, and also Hermitian, since it satisfies

$$\langle u, v\rangle_G = \overline{\langle v, u\rangle_G} \text{ and } \langle c_1u_1 + c_2u_2, v\rangle_G = c_1\langle u_1, v\rangle_G + c_2\langle u_2, v\rangle_G$$

It follows that if U is an invariant subspace of $\mathbb{C}^n$, then the orthogonal complement $U^\perp$ with respect to the G-invariant inner product, is also invariant via:

$$\langle u, w\rangle_G = 0 \text{ for all } u \in U \Rightarrow \langle u, hw\rangle_G = \langle h^{-1}u, w\rangle_G = 0 \text{ for all } u \in U \quad \square$$

By repeatedly applying this Proposition, we conclude that:

Corollary. If (V, ρ) is a finite-dimensional complex rep of a finite group G, then:

$$V = U_1 \oplus \cdots \oplus U_r$$

is a direct sum of irreducible invariant subspace representations.

We may couple this with:

Proposition 3. Each irreducible complex rep of an *abelian group* G is a character.

Proof. Let $\rho : G \to \mathrm{Aut}(\mathbb{C}^n)$. We show that the commuting matrices:

$$\rho(g) = A_g \text{ for } g \in G$$

all share a common eigenvector. The line spanned by one such eigenvector is then an invariant subspace for the representation ρ (and a character of the abelian group).

Select $g \in G$ and let $v \in \mathbb{C}^n$ be an eigenvector of A_g with eigenvalue $\lambda \in \mathbb{C}$. Select another $h \in G$. Then because A_g and A_h commute, we have:

$$A_g(A_hv) = A_h(A_gv) = A_h(\lambda \cdot v) = \lambda A_h(v)$$

so A_hv is **another** eigenvector for A_g with eigenvalue λ. View:

$$A_h : V_\lambda \to V_\lambda \text{ as a symmetry of the } \lambda\text{-eigenspace of } A_g$$

Then A_h has an eigenvector in V_λ with eigenvalue μ which is a shared eigenvector. Continue this process to conclude that any finite number of commuting matrices share an eigenvector. This also applies to an infinite number of commuting matrices acting on a finite-dimensional vector space, reasoning by induction on the dimension of the shared eigenspaces. $\square$

Example. Consider the "cycle" representation of C_n on $\mathbb{C}^n$ given by:

$$\rho(g)(e_i) = e_{i+1} \text{ for } i < n \text{ and } \rho(g)(e_n) = e_1$$

Then:

$$A_g = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

has a basis $v_1, ..., v_n$ of eigenvectors (hence invariant lines for C_n) given by:

$$v_m = e_1 + \omega_n^m e_2 + \omega_n^{2m} e_3 + \cdots + \omega_n^{(n-1)m} e_n \text{ (with eigenvalue } \omega_n^m\text{)}$$

and, in particular, $v_n = e_1 + \cdots + e_n$. Notice that each of the invariant lines:

$$(\langle v_m \rangle, \rho) \text{ is the character } \chi_m(g) = \omega_n^m$$

so that this representation is the direct sum of **all** the complex characters of C_n.

We combine the Propositions to get the following:

Corollary. Every finite dimensional complex representation of a finite abelian group is a direct sum of characters.

Remark. The subtle two-dimensional representation of $(\mathbb{Z}, +)$ above shows that finiteness of the abelian group is essential to the Corollary (and to Proposition 2), though Proposition 3 holds also for infinite abelian groups.

Corollary. If $A \in \text{Aut}(\mathbb{C}^n)$ and $A^d = I_n$, then A is semi-simple.

Proof. Since A semi-simple means that $\mathbb{C}^n$ has a basis consisting of eigenvectors of A, this is just a rewording of the previous Corollary for the representation of the cyclic group C_d given by $\rho(g) = A$.

Corollary. If $(\mathbb{C}^n, \rho)$ is a representation of a finite group G, then each

$$\rho(g) = A_g \text{ is semi-simple}$$

Proof. Each of these matrices has order d for some d. □

The Two-Dimensional Dihedral Representations. Let D_{2n} be the dihedral group, generated by two elements g_1 and g_2 with relations:

$$g_1^2 = 1, g_2^2 = 1 \text{ and } (g_1 g_2)^n = 1$$

One representation of D_{2n} is given by:

$$\rho(g_1) = \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ \sin(2\pi/n) & -\cos(2\pi/n) \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with

$$\rho(g_1 g_2) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}$$

Viewed as a real representation, these are the symmetries of a regular n-gon centered at the origin (with a vertex on the x-axis), generated by the reflection across the x-axis and the reflection across the line $y = \tan(\pi/n)x$. But using the same matrices, we may view this as a *complex* two-dimensional representation of D_{2n}.

Even as a two-dimensional complex representation, this is irreducible since any invariant line would be spanned by a common eigenvector for $\rho(g_1)$ and $\rho(g_2)$, and by virtue of being reflections across different lines of symmetry, they share no common complex eigenvectors.

Final Example. We've now seen two irreducible two-dim'l representations of S_3:

$$\rho(g_1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(g_2) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

and:

$$\tau(g_1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \text{ and } \tau(g_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

viewing S_3 as the dihedral group D_6 of symmetries of the equilateral triangle.

I claim that these are the **isomorphic** representation of S_3, with the different matrix representations an artifact of the choice of different bases for $\mathbb{C}^2$. In other words, we seek a single "change of basis" matrix B such that:

$$B^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

and

$$B^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is easiest to work with the second equation, and to recall that because the change of basis B converts to a *diagonal* matrix, then:

$$B = [v_1 \ v_2] \text{ where } \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} v_1 = v_1 \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} v_2 = -v_2$$

i.e. v_1 and v_2 are eigenvectors with $+1$ and -1 eigenvalues. A bit of fiddling gives:

$$v_1 = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } v_2 = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{C}^*$$

and then plugging in for B we find that setting $\lambda_2/\lambda_1 = \sqrt{3}$ gives

$$B^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$