HW #1 – MATH 6320
FALL 2024

DUE: MONDAY, JANUARY 22

1. Show that the image of a group homomorphism $f : G \to G'$ is a subgroup.

2. Let $H, H' \subset G$ be subgroups and let $HH' = \{hh' \mid h \in H, h' \in H'\} \subset G$.

(a) If $|H|$ and $|H'|$ are finite, show that:

$$|HH'| = (|H| \cdot |H'|)/(|H \cap H'|)$$

(b) Show with an example that $|HH'|$ need not divide $|G|$.

(c) On the other hand, show that if H or H' is normal, then $HH' \subset G$ is a subgroup, and so in that case $|HH'|$ does divide $|G|$.

(d) If $H' \subset G$ is normal, show that $H \cap H' \subset H$ is normal, and

$$|H/H \cap H'| \text{ divides } |G|/|H'|$$

3. Show that any subgroup $H \subset G$ with $2|H| = |G|$ is necessarily normal.

4. Prove that a group with 6 elements is isomorphic to either C_6 or S_3.

5. Prove that a group with 8 elements is isomorphic to one of:

$$C_8, C_4 \times C_2, C_2 \times C_2 \times C_2 \text{ (the abelian groups),}$$
$$D_8 \text{ (the dihedral group of symmetries of the square) or}$$
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \text{ with quaternionic multiplication}$$