HW #5 – MATH 6310
FALL 2022

DUE: FRIDAY, NOVEMBER 18

1. Suppose R is a commutative ring with 1 and:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

is a commutative diagram of R-modules with exact rows.

Consider the complex of *images*.

$$0 \to \mathrm{im}(a) \xrightarrow{f'} \mathrm{im}(b) \xrightarrow{g'} \mathrm{im}(c) \to 0$$

Prove that this need **not** be exact at the $\mathrm{im}(b)$ term with a counterexample.

2. Finish the proof of the snake lemma for:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

(as in #1) by showing that the complex:

$$0 \to \ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\delta} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c) \to 0$$

of R-modules is exact at the $\ker(c)$ and at the $\operatorname{coker}(a)$ terms.

3. Prove the Zigzag Lemma. That is, prove that given a short exact sequence of complexes of R-modules:

$$0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$$

the snake lemma produces the δ_i maps in a long exact sequence:

$$\cdots \to H_i(C_\bullet) \xrightarrow{f_i} H_i(C'_\bullet) \xrightarrow{g_i} H_i(C''_\bullet) \xrightarrow{\delta_i} H_{i-1}(C_\bullet) \to \cdots$$

of homology R-modules.

4. Given a right split short exact sequence of R-modules:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

with $h : C \to B$ satisfying $g \circ h = 1_C$, prove that the homomorphism:

$$f + g : A \oplus C \to B; \quad (f + g)(a, c) = f(a) + g(c)$$

is an R-module isomorphism.

5. Given a left split short exact sequence of R-modules:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

with $e : B \to A$ satisfying $e \circ f = 1_A$, prove that the morphism:

$$(e, g) : B \to A \oplus C; \quad (e, g)(b) = (e(b), g(b))$$

is an R-module isomorphism.

6. Prove that a short exact sequence may be right split (as in # 4) by some h if and only if it may be left split (as in #5) by some e.

7. Prove that $(\mathbb{Q}, +)$ is not a free abelian group.

Hint: Prove first that if it were free, it would have to be isomorphic to $\mathbb{Z}$ (i.e. a free abelian group of rank one).