Riemann Surfaces and Graphs
6. The Riemann Roch Theorem

The Riemann-Roch Theorem is a relation between two vector spaces associated to a divisor D of degree d on a Riemann surface S:

$$V(D) = \{\phi \in \mathbb{C}(S) - \{0\} \text{ with } \mathrm{div}(\phi) + D \geq 0\} \cup \{0\}$$

$$W(-D) = \{\omega \in \Omega[S] - \{0\} \text{ with } \mathrm{div}(\omega) - D \geq 0\} \cup \{0\}$$

(recall that $\mathrm{div}(\omega)$ is an effective divisor of degree $2g-2$ for $\omega \in \Omega[S]$).

Choose a meromorphic differential $\tau \in \Omega(S)$, and set:

$$K_S = \mathrm{div}(\tau)$$

This is called a "canonical" divisor of degree $2g-2$, though it isn't actually canonical. On the other hand, once τ is chosen, then the vector spaces $V(K_S - D)$ and $W(-D)$ are isomorphic via the map $\phi \mapsto \phi \cdot \tau$ so we will follow the literature and replace $W(-D)$ with $V(K_S - D)$, keeping in mind that it is really $W(-D)$ that we want to work with.

Note. As we saw earlier, graphs **do** have truly canonical divisors.

Riemann-Roch. The dimensions of $V(D)$ and $V(K_S - D)$ satisfy:

$$\dim(V(D)) - \dim(V(K_S - D)) = \deg(D) + 1 - g$$

where g is the genus of the Riemann surface S.

Half of the Proof. We begin by proving the inequality:

$$\dim(V(D)) - \dim(V(K_S - D)) \leq \deg(D) + 1 - g$$

To do this, we will assume that $V(D) \neq 0$, and replace D (if necessary) with a linearly equivalent *effective* divisor $E = \sum_{i=1}^n e_i p_i$ of degree d. Then $V(D) \cong V(E)$ and $V(K_S - D) \cong V(K_S - E)$.

Next we introduce the **vector space of Laurent tails**, which we define (non-canonically) by choosing a local coordinate z_i near p_i with $z_i = 0$ at the point p_i and then setting $\mathrm{Laur}(E) = \{a_{i,-e_i} z_i^{-e_i} + \cdots + a_{i,-1} z_i^{-1} \mid a_{i,j} \in \mathbb{C}\}$, a vector space of dimension d. We are interested in two maps:

(i) The "tail" map:

$$\lambda : V(E) \to \mathrm{Laur}(E)$$

expanding ϕ as a Laurent series in the variables z_i and truncating, and:

(ii) The (locally defined) "residue" pairing:

$$\mathrm{Laur}(E) \times \Omega[S] \to \mathbb{C}$$

expanding $\omega = \psi(z_i)dz_i \in \Omega[S]$ around each point p_i, multiplying by the Laurent tail, "reading" off the coefficients of $z_i^{-1}dz_i$, and taking their sum. This defines a linear map:

$$\rho : \mathrm{Laur}(E) \to \Omega[S]^*$$

(a) The kernel of λ is the vector space of constant functions.

(b) The image of ρ is the kernel of the map $\Omega[S]^* \to W(E)^*$, and

(c) the **composition** $\rho \circ \lambda$ is the zero map.

In other words:

$$0 \to \mathbb{C} \to V(E) \to \mathrm{Laur}(E) \to \Omega[S]^* \to V(K_S - E)^* \to 0$$

is a complex of vector spaces that is exact everywhere except possibly at the middle term, and then it follows that:

$$1 - \dim(V(E)) + d - g + \dim(V(K_S - E) \geq 0$$

which is the desired Riemann-Roch inequality.

Our next goal is to prove that this sequence is, in fact, exact.

Your Assignment. Read this and make sense of it. Then:

1. Find the isomorphisms $V(D) \cong V(E)$ and $V(K_S - D) \cong V(K_S - E)$ that are asserted above to exist.

2. Prove (a) above.

3. Keeping in mind that the map $\Omega[S]^* \to W(-E)^*$ is dual to the inclusion map of vector spaces, prove (b).

4. Prove (c).

5. Prove the inequality directly when $V(D) = 0$.