What about the Riemann-Roch Theorem in general? For example, given an effective divisor D of positive degree on a Riemann surface, we have:

$$\dim(V(K_S + D)) \leq g + \deg(D) - 1$$

by the Riemann-Roch inequality, but Corollary 6.2 does not apply since $-D = K_S - (K_S + D)$ is not linearly equivalent to an effective divisor.

This particular example is quite important, since equality gives exactness of an analogue of the exact sequence in Proposition 6.1(c) for meromorphic **differentials** (as opposed to a meromorphic functions):

$$0 \to W(0) = \Omega[S] \to W(D) \to \text{Laur}(D) \to \mathbb{C} \to 0$$

which, in terms of local coordinates z_i around the points p_i appearing in D, maps a meromorphic differential to its Laurent tail and maps a Laurent tail to its "residue," namely the sum of the coefficients $a_{i,-1}$ of each z_i^{-1}. Riemann-Roch in this context is used in the proof of Abel's Theorem.

We will prove the Riemann-Roch Theorem with a more sophisticated version of Corollary 6.2 by considering Riemann surfaces "in the wild," i.e. embedded in projective space. Note that by Proposition 6.6., every Riemann surface of genus $g \geq 2$ that is not hyperelliptic embeds in $\mathbb{CP}^{g-1}$. Before we proceed let's tackle the hyperelliptic Riemann surfaces.

Hyperelliptic Riemann Surfaces in $\mathbb{CP}^1 \times \mathbb{CP}^1$. We can *almost* describe a hyperelliptic Riemann surface as a **complex curve** in $\mathbb{C}^2$ via an equation:

$$C = \{(x, y) \in \mathbb{C}^2 \mid y^2\alpha(x) + y\beta(x) + \gamma(x) = 0\}$$

of degree two in the y variable and degree $d = \max(\deg(\alpha, \beta, \gamma))$ in x. We will assume that α, β and γ all have degree d with no multiple roots and no shared roots. Each fiber $\pi^{-1}(x_0)$ of the projection $\pi : C \to \mathbb{C}$ to the x-axis is the set of zeroes of the polynomial:

$$f(x_0, y) = y^2\alpha(x_0) + y\beta(x_0) + \gamma(x_0) \text{ which is:}$$

(i) Two distinct points if $\alpha(x_0) \neq 0$ and $\Delta(x_0) \neq 0$, where

$$\Delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x)$$

is the discrminant, which we'll assume to also be of maximal degree $2d$.

(ii) One (ramified) point if $\Delta(x_0) = 0$ but $\alpha(x_0) \neq 0$

(iii) One non-ramified point if $\alpha(x_0) = 0$ but $\Delta(x_0) = \beta(x_0)^2 \neq 0$.

The condition for C to be a complex manifold is:

$$f(x,y) \neq 0 \;\text{ or }\; \frac{\partial f}{\partial x}(x,y) \neq 0 \;\text{ or }\; \frac{\partial f}{\partial y}(x,y) \neq 0$$

for all points $(x,y) \in \mathbb{C}^2$. But $f(x_0,y_0) = 0 = (\partial f/\partial y)(x_0,y_0)$ if and only if:

$$\Delta(x_0) = 0 \;\text{ and }\; y_0 = -\frac{\beta(x_0)}{2\alpha(x_0)}$$

and the additional condition $(\partial f/\partial x)(x_0,y_0) = 0$ is equivalent to $\Delta'(x_0) = 0$.

Thus for $f(x,y)$ to define a Riemann surface, one only needs to be sure that the discriminant polynomial $\Delta(x)$ has no multiple roots. Next, we introduce a z variable to "homogenize the y variable" in $f(x,y)$ giving:

$$y^2\alpha(x) + yz\beta(x) + z^2\gamma(x)$$

the zeroes of which, in $\mathbb{C} \times \mathbb{CP}^1$ add d points to C, namely the "missing" points of the projection map over the roots of $\alpha(x)$. Finally, we introduce a w-variable to homogenize the x variable, replacing:

$$\alpha(x) \text{ by } A(x,w) = w^d \cdot \alpha(x/w), \text{ etc}$$

or equivalently, if $\alpha(x) = a(x-r_1)\cdots(x-r_d)$, then

$$A(x,w) = a(x-r_1w)\cdots(x-r_dw)$$

This further enlarges C, adding two more points and completing it to a closed Riemann surface S embedded in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Notice that:

$$\pi^*(x) = \phi \text{ is the meromorphic function on } S \text{ with two poles}$$

defining S as a hyperelliptic Riemann surface, and:

$$\rho^*(y) = \psi \text{ is another meromorphic function on } S \text{ with } d \text{ poles}$$

where ρ is the "other" projection to the y-axis (extended to $\mathbb{CP}^1$).

Proposition 6.7. The genus of S is $d-1$.

Proof. By the Riemann-Hurwitz formula, since $\pi : S \to \mathbb{CP}^1$ ramifies only over the $2d$ zeroes of the discriminant $\Delta(x)$, we get:

$$2g - 2 = 2(-2) + \sum(e_p - 1) = -4 + 2d \;\text{ and }\; g = d - 1 \quad \square$$

In particular, the "extra" meromorphic function $\rho^*(y)$ has $g+1$ poles.

Proposition 6.8. **Every** hyperelliptic S is isomorphic to one of these.

Proof. If S, S' are hyperelliptic Riemann surfaces of genus g with maps:

$$\phi : S \to \mathbb{CP}^1 \quad \text{and} \quad \psi : S' \to \mathbb{CP}^1$$

of degree two, ramified over the **same** set of points $x_1, ..., x_{2g+2} \in \mathbb{CP}^1$, then $S \cong S'$. Thus, to prove that our construction gives all hyperelliptic curves, we need to simply find, given distinct complex numbers $x_1, ..., x_{2g+2} \in \mathbb{C}$, three polynomials $\alpha(x), \beta(x)$ and $\gamma(x)$ each of degree $d = g+1$ such that:

$$\Delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x) = c(x - x_1)\cdots(x - x_{2g+2})$$

This is left to the reader as an exercise.

A Final Remark. By composing with the further embedding:

$$S \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$$

of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$ as a quadric surface, we see that every hyperelliptic curve embeds in $\mathbb{CP}^3$. Indeed, **every** Riemann surface embeds in $\mathbb{CP}^3$.

Assignment. 1. Read and complain if you don't understand something.

2. Find a graphing calculator (e.g. desmos.com) and play with equations:

$$y^2\alpha(x) + y\beta(x) + \gamma(x) = 0 \text{ of your choosing}$$

The curves you get can be quite intricate. Share your most inspired creations with me, and I'll forward them to the class.

For example, explain the features of the curve:

$$y^2(x^2 - 1) + y(x^2 - 6) + (x^2 - 9) = 0$$

(Keep in mind that we can't see all the features of the complex solutions in this set of real solutions.)

3. Tackle the Exercise in the final Proposition.