Riemann Surfaces and Graphs
1. Chip Firing on Graphs

Let Γ be a connected graph with no loops.

Firing 1.0. *Firing* from a vertex $v_1 \in V$ converts a divisor $D_0 = \sum d_v v$ to:

$$D_1 = (d_{v_1} - val(v_1))v_1 + \sum_{\text{neighbors of } v_1} (d_w + 1)w + \sum_{\text{other vertices}} d_v v$$

i.e. thinking of d_v as the number of dollars owned (or owed) by v, firing sends one dollar from v_1 to each neighboring vertex. Then $\deg(D_0) = \deg(D_1)$, i.e. the total number of dollars owned (or owed) is unchanged by firing.

Definition 1.1. Divisors D, D' (of the same degree) are *chip-related* if there is a sequence of chip firings from vertices $v_1, v_2, ..., v_n \in V$ giving:

$$D = D_0 \overset{v_1}{\rightsquigarrow} D_1 \rightsquigarrow \cdots \overset{v_n}{\rightsquigarrow} D_n = D'$$

Example. Let Γ be the triangle with adjacent vertices abc. Then

$$2a \overset{a}{\rightsquigarrow} b + c \overset{b}{\rightsquigarrow} a - b + 2c \overset{c}{\rightsquigarrow} 2a$$

Lemma 1.2. Being chip-related is an equivalence relation (denoted by $\sim$).

Proof. If $S \subset V$, then firing once from all vertices of S (in any order) has the net effect of passing one dollar across each edge joining S to S^c. In particular, firing once from all the vertices of V (as in the example above) returns each divisor to itself, so the chip-relation is reflexive. This also shows that chip-firing from v has an inverse, namely firing once from each vertex in $V - \{v\}$, and it follows that the chip-relation is symmetric. Finally, transitivity is immediate from the definition.

Let $D \in \mathbb{Z}[V]$ be a divisor of degree d (not necessarily effective).

Definition 1.3. $|D| \subset V_d$ is the set of *effective* divisors in the equivalence class of D. In particular, $|D| = \emptyset$ if D is not chip-related to any effective divisors (e.g. in the case $d < 0$).

The Game. Given a divisor D of degree ≥ 0, try to create an effective divisor by chip-firing. Then find the entire equivalence class $|D|$ and use this information to distinguish graphs. In the context of Riemann Surfaces, the analogues of $|D|$ are the (complete) *linear series.* We will see that these equivalence classes are also "linear" by making use of the *tropical numbers.*

Example. Let $\Gamma = \{a, b, c, d, e, f\}$ be the (adjacent) vertices of a hexagon. Then $|2a| = \{2a, b+f, c+e, 2d\} = |2a|$, but the equivalence $b+f \sim c+e$ requires the firing of the three vertices a, b and f!

Proposition 1.4. If $e \in E$ is a *bridge*, i.e. removing e disconnects Γ, then the ends p and q of the edge e satisfy $p \sim q$.

Proof. Let $p \in S$ and $q \in S^c$ be the vertices of the two connected components of $\Gamma - \{e\}$. Then firing from all the vertices of S sends exactly one dollar across the vertex e from p to q. □

Corollary 1.5. (a) If $g(\Gamma) = 0$, then all vertices are equivalent as divisors.

(b) If $g(\Gamma) = 0$, then all divisors of any degree are equivalent.

Proof. Each edge of a tree is a bridge. This gives (a). For (b), we use the fact that if $D_0 \overset{v}{\rightsquigarrow} D_1$, then $D_0 + E \overset{v}{\rightsquigarrow} D_1 + E$ for any divisor E, hence:

$$D \sim D' \Rightarrow D + E \sim D' + E$$

for any divisor E, and then (b) follows immediately from (a).

Next, we prove a converse to Proposition 1.4 using the following:

Lemma 1.6. If $D, D' \in V_d$ and $D \sim D'$, then there is a chain of subsets $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m \subset V$ such that firing from the vertices of S_i:

$$D = D_0 \overset{S_1}{\rightsquigarrow} D_1 \overset{S_2}{\rightsquigarrow} D_2 \cdots \rightsquigarrow D_m = D'$$

gives a sequence of equivalent **effective** divisors $D_i \in V_d$.

Proof. Let the equivalence $D \sim D'$ be given by chip-firings of vertices: $v_1, ..., v_n$ and sum the vertices to create an effective divisor:

$$C_1 = v_1 + \cdots + v_n = \sum_{v \in V} c_v v$$

(note that chip-firing commutes!). Now let $c = \max\{c_v\}$ and let:

$$S_1 = \{v \mid c_v = c\}$$

Next let $C_2 = C_1 - \sum_{v \in S_1} v$, define S_2 analogously, and proceed by induction. This produces an increasing chain of subsets of V that accounts for the all the chip firings of the equivalence relation. Thus we do get a sequence of chip firings and equivalent divisors as above.

If D_{i-1} is effective but some vertex v has a negative coefficient in D_i, then necessarily $v \in S_i$, because the vertices of S_i are the "givers" of dollars. But then v is a "giver" vertex in all subsequent firings, since the sets are nested, so v never has an opportunity to "take" a dollar and return to a positive coefficient. Since the final divisor D' is assumed to be effective, it follows that there is no such vertex v. □

Proposition 1.7. If $p \sim q$, then there is a path of bridges from p to q.

Proof. Let $p = p_0 \overset{S_1}{\rightsquigarrow} p_1 \overset{S_2}{\rightsquigarrow} \cdots \rightsquigarrow p_n = q$ be the sequence of chip firings from the Proposition. (An effective divisor of degree one is a vertex!) Then out of each set S_i there must be a **single** edge joining S_i to S_i^c, passing from p_{i-1} to p_i. That is, the edge with $\epsilon(e) = p_{i-1} + p_i$ must be a bridge. □

Corollary 1.8. (a) The (distinct) vertices of a *circuit* are not equivalent.

(b) The vertex set of the "debridgification" graph Γ_{deb} obtained from Γ by collapsing all the bridges is in a natural bijection with the equivalence classes $|p|$ of the vertices of Γ. Notice that $g(\Gamma_{deb}) = g(\Gamma)$, since collapsing each bridge reduces both the vertex set and the edge set by one element.

A *leaf* is a bridge from a singleton (valence one) vertex to the rest of Γ. The "deleafification" Γ_{del} is a partial debridgification obtained by collapsing only leaves, resulting in a graph whose vertices all have valence 2 or more.

Examples. (i) If $g(\Gamma) = 0$, then $\Gamma_{del} = \Gamma_{deb}$ is a singleton vertex.

(ii) If $g(\Gamma) = 1$, then Γ_{del} has **only** vertices of valence two, and it follows that $\Gamma_{del} = \Gamma_{deb}$ is a polygon P_n with n vertices and n edges.

(iii) Joining a polygon P_m with a polygon P_n via a bridge from one vertex of P_m to a vertex of P_n gives a genus two graph with a bridge and no leaves.

Next, we define the *rank* $r(D)$ of a divisor D.

Definition 1.9. $r(D) = -1$ if $|D| = \emptyset$. Otherwise, $r(D) \geq 0$ is the maximal integer so that for all $E \in V_r$, there is an $F \in V_{d-r}$ with $E + F \in |D|$.

Examples. (a) If D is a divisor of degree d on a **tree**, then $|D| = V_d$ is the set of **all** effective divisors of degree d, so $r(D) = d$

(b) Let Γ be a (genus one) polygon P_n. Then:

- $r(a - b) = -1$ if $a \neq b$ by Corollary 1.8 (a) above, but $r(0) = 0$.
- $r(v) = 0$ for all $v \in V(P_n)$, since $|v| = \{v\}$ by Corollary 1.8 (a).

Exercise. Show that the 0 divisor on P_n is "special" in the sense that $r(0) = 0$, but $r(D) = d - 1$ for **all other** divisors of all degrees $d \geq 0$.

Definition 1.10. The *canonical divisor* of a graph Γ is the divisor:

$$K_\Gamma = \sum_{v \in V} (val(v) - 2)v$$

Remark. $\deg(K_\Gamma) = 2(\#E - \#V) = 2g(\Gamma) - 2$ by the genus formula.

Example. (a) $K_{P_n} = 0$ is the special divisor on a polygon.

(b) Consider the complete graph K_4 on 4 vertices v_1, v_2, v_3, v_4, Then:

$$K_\Gamma = v_1 + v_2 + v_3 + v_4 \text{ and}$$

$$|K_\Gamma| = \{v_1 + v_2 + v_3 + v_4, 4v_i\} \text{ has rank } 2 = g - 1$$

Exercise. Catalog the other divisor classes $|D| \subset V_4$ for the graph K_4 and check that each of them has rank 1.

We will eventually prove:

The Riemann-Roch Theorem (for graphs). For any divisor D on Γ,

$$r(D) - r(K_\Gamma - D) = \deg(D) + 1 - g(\Gamma)$$

One simple consequence of this is:

(i) $r(K_\Gamma) = g(\Gamma) - 1$ and

(ii) $r(D) = g(\Gamma) - 2$ if $\deg(D) = 2g(\Gamma) - 2$ and $|D| \neq |K_\Gamma|$.

Definition 1.11. $D \in V_d$ is **special** if $r(D) \geq 0$ and $r(K_\Gamma - D) \geq 0$.

Definition 1.12. If either (a) $r(D) = r(D - p)$ or (b) $r(D + p) = r(D) + 1$ then $|D|$ is a *redundant* divisor class, since its rank is determined by the rank of a "more special" divisor class.

The irredundant divisor classes $|D|$ of special divisors are the "signature" of a graph, distinguishing it from other graphs of the same genus. In the same way, the special divisors will be the "signature" of a Riemann Surface.

Examples. Consider the following "test" graphs:

- The complete bipartite graphs $K_{m,n}$ of genus $(m - 1)(n - 1)$.
- The complete graphs K_d of genus $(d - 1)(d - 2)/2$.

Exercise. If D is a special divisor, then $0 \leq \deg(D) \leq 2g - 2$. and $|D|$ is special (and irredundant) if and only if $|K_\Gamma - D|$ is special (and irredundant).

Let's consider the first few genera:

genus zero There are no special divisors in genus zero.

genus one The zero divisor is special in all genera $g \geq 0$. In genus one, $|0|$ is the only special divisor class (and it is irredundant).

Each $K_{m,n}$ has two natural divisor classes, namely:

$$|v_1 + \cdots + v_m| \text{ and } |w_1 + \cdots + w_n|$$

where the v_i and w_j are the vertices of the two sets of the bipartite graph. Then:

$$r(v_1 + \cdots + v_m) = 1 = r(w_1 + \cdots + w_n)$$

and each v_i has valence n and each w_j has valence m.

The *hyperelliptic* complete bipartite graphs $K_{2,n}$ satisfy $n = g - 1$ and $K_\Gamma = (g-1)(v_1+v_2)$. The divisor classes $|k(v_1+v_2)|$ for $k = 0, ..., g-1$ have rank k and are **all** the special irredundant divisor classes (sids) on $\Gamma = K_{2,n}$

genus two For graphs of genus $g \geq 1$, there are no sids of degree one, by Corollary 1.8. Thus the only sid classes on a graph Γ of genus two are $|0|$ and $|K_\Gamma|$. In particular, when $\Gamma = K_{2,3}$, we have $K_\Gamma = v_1 + v_2$ and the class $|w_1 + w_2 + w_3|$ is not special, but it is irredundant.

genus three Consider the two graphs $K_{2,4}$ and K_4. Then:

- The sids of degree ≤ 3 on $\Gamma_{2,4}$ are $|0|$ and $|v_1 + v_2|$ with

$$|K_\Gamma - (v_1 + v_2)| = |v_1 + v_2|$$

- The **only** sid classes on K_4 are $|0|$ and $|K_\Gamma|$.

genus four

- The graph $K_{2,5}$ has sids $|0|, |v_1 + v_2|$ in the degree range $[0, 3]$ and sids $|K_\Gamma|$ and $|K_\Gamma - (v_1 + v_2)| = |2(v_1 + v_2)|$ in the range $[4, 6]$.

- The graph $K_{3,3}$ has no sids of degree two, but:

$$|v_1 + v_2 + v_3| \neq |w_1 + w_2 + w_3|$$

are distinct sid classes of rank one, with $K - (v_1 + v_2 + v_3) = (w_1 + w_2 + w_3)$. These (together with 0 and K_Γ) are the only sids on $K_{3,3}$.

Exercise. genus six. Find the sids for $K_{2,7}, K_{3,4}$ and K_5.

Next, we introduce a very powerful idea for playing the game:

Definition 1.13. Let Γ be a graph, and fix $q \in V$. Then a divisor:

$$D = E + nq;\ n \in \mathbb{Z}$$

is *q-reduced* if E is effective and firing from every subset $S \subset V - \{q\}$ puts some vertex (other than q) in a deficit.

These divisors are a useful tool for analyzing divisor classes because of:

Dhar's Burning Algorithm. (i) For each $q \in V$, every divisor $D \in \mathbb{Z}[V]$ is equivalent to a *unique* q-reduced divisor $D_q \in \mathbb{Z}[V]$.

(ii) A divisor $E + nq$ with E effective can be checked for q-reducibility by the following **burning algorithm**:

Think of the chips of E as firefighters. Light a fire at q and let it spread through the graph. If a vertex p is occupied by fewer firefighters than incoming fires, they are overwhelmed and the fire spreads to p. The fire eventually spreads to the entire graph if and only if D is q-reduced.

Proof. Given q, the *distance* $d(p, q)$ from q to $p \in V$ is the length of the shortest path from q to p. Given D, fire chips from q until each vertex $p \in V$ with $d(p, q) = 1$ is out of debt. Then fire additional chips until each such vertex has at least $val(p)$ chips. At that point, we may fire from all the vertices p without putting any of them into debt. Then we return to q and fire again until we can fire a second time from each vertex p, eventually putting all the vertices o with $d(q, o) = 2$ out of debt. We continue in this vein until we obtain a divisor of the form $E + nq$ with E effective.

Next, either $E + nq$ is q-reduced, or else there is a subset $S \subset V - \{q\}$ from which may fire to obtain a new divisor $E' + nq$. If there is $p \in S$ with $d(q, p) = 1$, then this increases the coefficient of q and in that sense "improves" the situation. If not, we need another measure to see that E' is an improvement on E. To this end, we define a "word" associated to the effective divisor $E = \sum_{p \neq q} e_p p$:

$$w(E) = (d_1, ..., d_m) \text{ where } d_i = \sum_{d(q,p)=i} e_p$$

with the lexicographic ordering (in which, e.g. $(3, 0, 0) > (2, 1, 0) > (2, 0, 1)$) Then if $d(q, p) > 1$ for all $p \in S$, the new divisor E' satisfies $w(E') > w(E)$ in the lexicographic ordering since the vertices $p \in S$ minimizing the distance $d(q, p)$ will fire chips into vertices that are closer to q!

To prove uniqueness, let $E + nq \sim E' + mq$ be a non-trivial equivalence realized via a sequence of chip firings and as in Lemma 1.6, accumulate them into a divisor $C = \sum c_v v$ with $c = \max_v\{c_v\}$ and set $S = \{v \mid c_v = c\}$. Then as in Lemma 1.6, firing from the vertices of S cannot put any vertex of S into debt. Thus, if $q \notin S$, it follows that $E + nq$ is not q-reduced. On the other hand, if $q \in S$, then we use the *complementary* chain of subsets $S_n^c \subset \cdots \subset S_1^c$ (realizing the reverse chip firings) to conclude that $q \notin S_n^c$ and therefore that $E' + mq$ is not q-reduced! This concludes the proof of (i).

As for (ii), if $E + nq$ is not q-reduced, let $S \subset V - \{q\}$ be a set from which firing produces a new divisor $E' + mq$. Then the firefighters on the perimeter of S (the set of vertices with edges connecting S to S^c) prevent the forest fire from lighting the vertices of S. On the other hand, if the forest fire fails to burn the full graph, then the unburnt vertices must comprise a set S from which firing exhibits $E + nq$ as a non-q-reduced divisor. □

Corollary 1.14. $|D| \neq \emptyset$ if and only if D_q is effective for **all** $q \in V$.

Proof. If D_q is effective, then $D \sim D_q$, so $|D| \neq \emptyset$. On the other hand, if $D \sim D'$ and D' is effective, then for each $q \in V$, either $D' = E + nq$ is q-reduced, or else firing from some $S \subset V - \{q\}$ produces $D'' = E' + n'q$, with E' effective and $n' \geq n$ (since q is a "taker"). In particular, D_q is obtained by a finite number of such chip-firings, so it is effective. □

Project. Use this to design an efficient algorithm for finding $r(D)$.

For each *orientation* $\mathcal{O}$ of a finite connected graph Γ, let:

$$D_{\mathcal{O}} = \sum_{v \in V} (val_{in}(v) - 1)v$$

where $val_{in}(v)$ is the incoming valence (of edges pointing to v). Then:

$$\deg(D_{\mathcal{O}}) = g - 1 \text{ and } D_{\mathcal{O}} + D_{\mathcal{O}^-} = K_\Gamma$$

where $\mathcal{O}^-$ is the opposite orientation. Note that $D_{\mathcal{O}}$ is an effective divisor away from the *sources* of $\mathcal{O}$, i.e. the vertices with only outgoing edges, and $D_{\mathcal{O}^-}$ is effective away from the *sinks* of $\mathcal{O}$ (= sources of $\mathcal{O}^-$).

Definition 1.15. $\mathcal{O}$ is *acyclic* if Γ has no oriented circuits.

Note that $\mathcal{O}$ is acyclic if and only if $\mathcal{O}^-$ is acyclic.

Construction. Number the vertices V of a graph with no loops, and orient each edge so that it points from v_i to v_j if $i < j$. This is an acyclic orientation with v_1 as a source and v_n as a sink (but there can be more, of course!).

Lemma 1.16. An acyclic orientation has both sources and sinks, i.e. both the divisors $D_{\mathcal{O}}$ and $D_{\mathcal{O}^-}$ fail to be effective when $\mathcal{O}$ is acyclic.

Proof. Start from some vertex v and construct a path consisting of an arbitrarily chosen outward pointing edge from each vertex. If $\mathcal{O}$ has no sources and is acyclic, then this path never visits the same vertex twice, and therefore continues indefinitely, contradicting the finiteness of Γ. □

Let $D_q = nq + E$ be a q-reduced divisor and orient the edges of the graph Γ outward with the spread of the all-consuming fire from the burning algorithm. If the fire reaches a set S of vertices simultaneously, orient the subgraph spanned by S as in the construction. Then $\mathcal{O}$ satisfies:

(a) $\mathcal{O}$ is acyclic.

(b) The vertex q is the unique source.

(c) The divisor $D_{\mathcal{O}} = -q + E_{\mathcal{O}}$ is q-reduced (and $\deg(E_{\mathcal{O}}) = g$).
In particular, $r(D_{\mathcal{O}}) = -1$.

(d) $E_{\mathcal{O}} - E$ is an effective divisor.

As an immediate Corollary, we get:

Jacobi Inversion Theorem. If $\deg(D) \geq g$, then $|D| \neq \emptyset$.

Proof. Pick $q \in V$ and let $D_q = nq + E$ be the unique q-reduced divisor equivalent to D. From (c) and (d), $\deg(E) \leq g$, so $n \geq 0$ and D_q is effective!

Of course, it follows that if $\deg(D) \geq g + r$, then $r(D) \geq r$.