6/30-29

$S_\bullet = k[x,y,z]$

$S_d(p_1, \ldots, p_m) \subseteq S_d$

Proposition:

if $\dim S_3(p_1, \ldots, p_8) \geq 3$

then: (1) 4 of the pts. are collinear

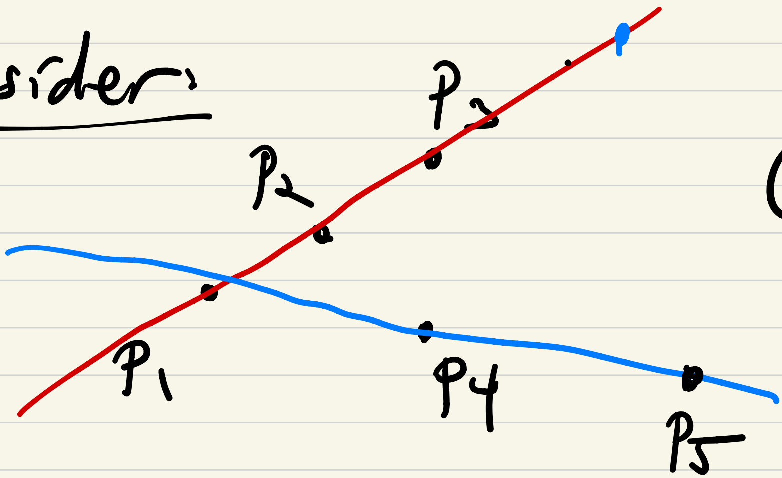
or

(2) 7 of the pts lie on an irreducible conic.

Warmup:

$\dim S_2(p_1, \ldots, p_5) \geq 2$, then 4 of the points are collinear

Consider:



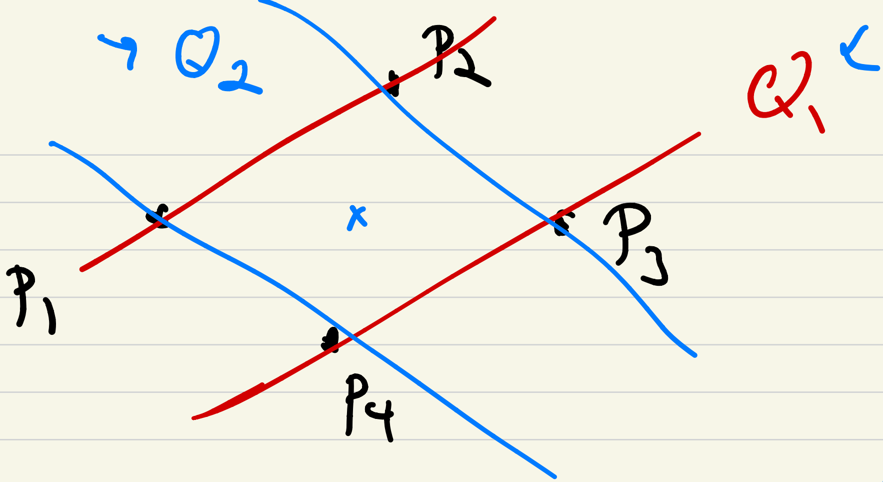
(1)

$S_2(p_1, p_2, p_3, p_4) = L \cdot S_1(p_4)$

(dim = 2)

If $p_5 \notin X(L)$, then

$S_2(p_1, \ldots, p_5) = L \cdot S_1(p_4, p_5)$



(2)

$$S_2(P_1, \dots, P_4) = \underline{\underline{\lambda_1 Q_1 + \lambda_2 Q_2}}$$

$\forall P_5 \in \mathbb{P}^2 \setminus (P_1, \dots, P_4)$

either $Q_1(P_5) \neq 0$ or $Q_2(P_5) \neq 0$

$$\Longrightarrow \underset{\underline{\underline{\dim = 1}}}{S_2(P_1, \dots, P_5)} \subsetneq \underset{\dim = 2}{S_2(P_1, \dots, P_4)}$$

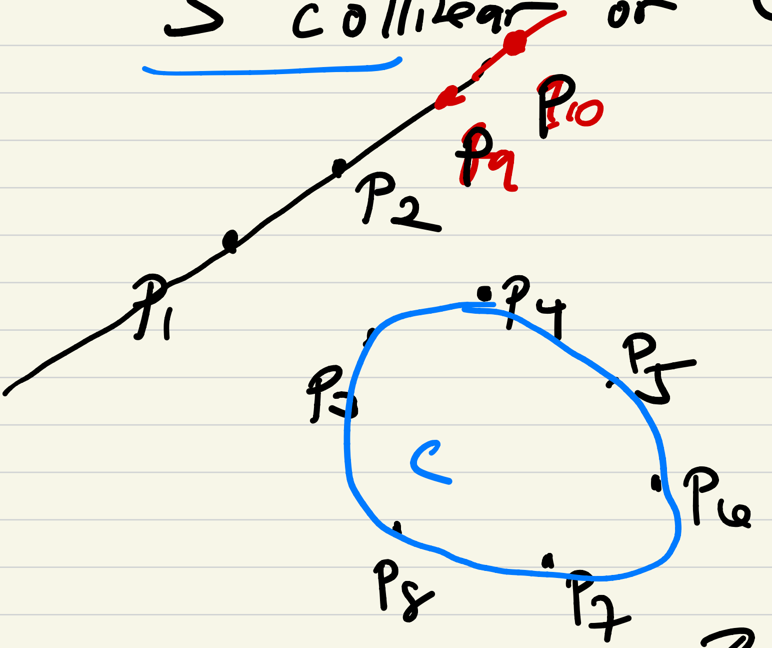
Prop: $\dim S_3(p_1, \ldots, p_8) \geq 3$

$\Rightarrow$ 4 collinear / 7 on a conic.

Pf:

(1) $\dim S_3(p_1, \ldots, p_8) \geq 3 \Rightarrow$

3 collinear or 6 lie on a conic



$\dim \geq 3 \Rightarrow \dim S_3(p_1, \ldots, p_{10}) \geq 1 \Rightarrow F$

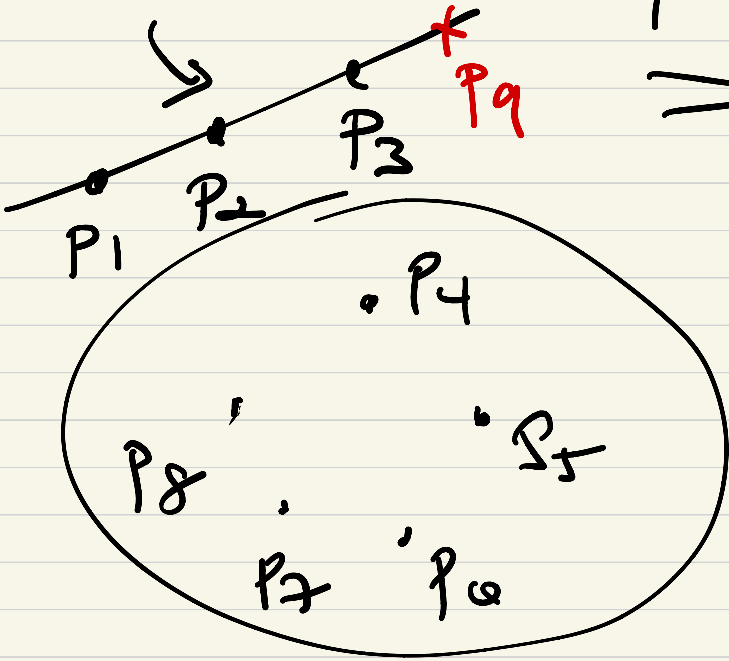
$$\Rightarrow \quad F = \underset{\substack{\uparrow \\ \text{line through} \\ p_1, p_2, p_9, p_{10}}}{L} \cdot Q$$

$$\Rightarrow \quad \underbrace{p_3, \dots, p_8}_{\text{6 of them!}} \in X(Q)$$

Can
Assume 3 on a line or
6 on a conic

dim ≥ 3 and

(2) Assume P_1, P_2, P_3 are collinear. Conclude that 4 collinear



If $\dim S_3(p_1, \ldots, p_8) \geq 3$, then $\dim S_3(p_1, \ldots, p_9) \geq 2$. But $S_3(p_1, \ldots, p_9) = L \cdot S_2(p_4, p_5, p_6, p_7, p_8)$ $\dim \geq 2$

$\Rightarrow$ 4 of $P_4, \dots, P_8$

lie on a line!

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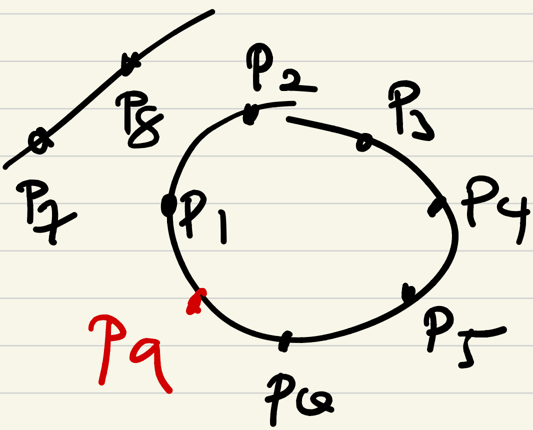
(2) Assume $\dim \geq 3$ and

$P_1, \dots, P_6$ lie on a conic.

$\Rightarrow$ 7 lie on

a conic.



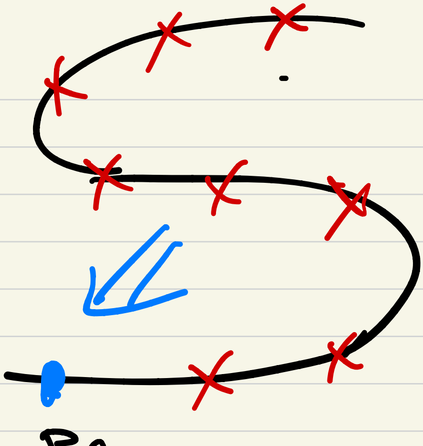
Otherwise:

$\dim \geq 2$ $\lightning$ $\boxed{\dim = 1}$

Conclude: $S_3(P_1, \dots, P_9)$

$= Q \cdot S_1(P_7, P_8) = \lambda Q \cdot L$

Cor:



cubic

(med.)

$X(F)$

Any set of 8 pts on an irred cubic satisfy

$\dim S_3(P_1, \dots, P_8) = 2$ ✓

(Because No 4 on a line
No 7 on a conic)

Let $S_3(P_1, \dots, P_8) = \{\lambda F + \mu G\}$

~~then~~ and P_9 is the ninth pt. of $X(F) \cap X(G)$

Any other $H \in S_3(p_1, \dots, p_8)$

is of the form

$$\lambda F + \mu G, \text{ so}$$

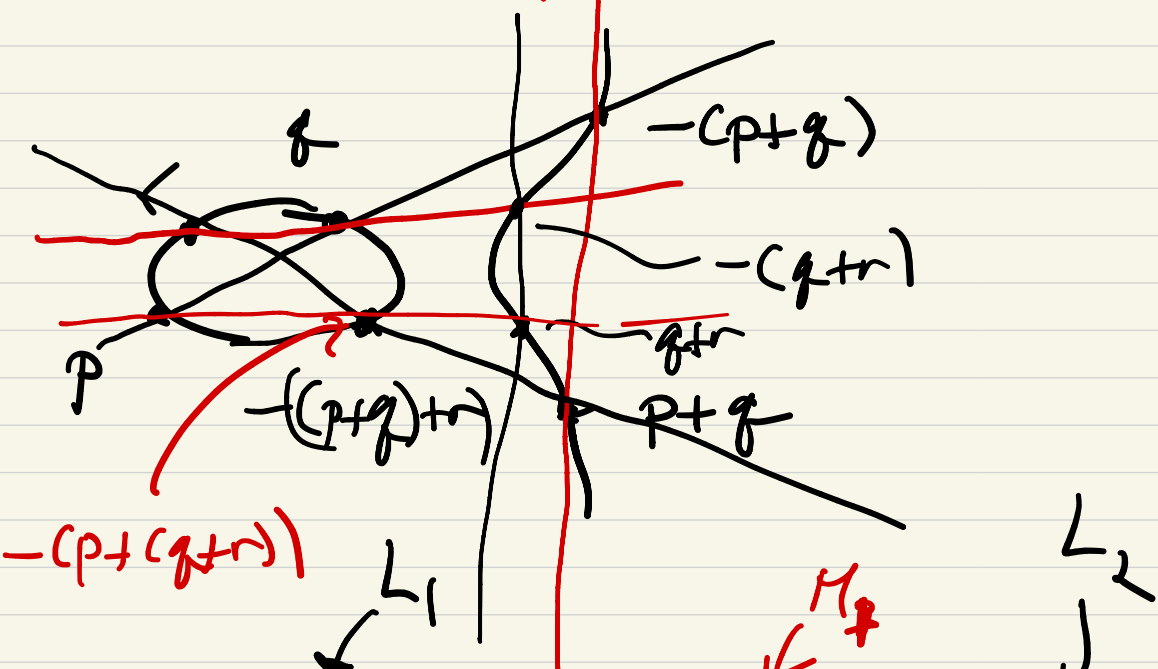
$X(H) \ni p_9$ as well !

A

Associativity of $\oplus$ on cubic

$p, q, r \in C \implies p \oplus (q \oplus r) = (p \oplus q) + r$

$[0:1:0] = e$



$p, q, -(p+q)$, $-(p+q), p+q, e$, $p+q, r,$

M_2 $\quad$ L_3 $\quad$ $-(p+q)+r$

$q, r, -(q+r)$, $-(q+r), e, q+r$, $p, q+r,$ M_3

$-(p+(q+r))$

Upshot:

$X(L_1 L_2 L_3) \cap C$

and $X(M_1 M_2 M_3) \cap C$

share 8 pts, so they must also share the 9^{th}!

Assumption: No tangent lines in either construction.

$$E \times E \times E \underset{g}{\overset{f}{\rightrightarrows}} E$$

$$f: -(p+(q+r)), \quad g: -((p+q)+r)$$

$f = g$ on an open subset of $E \times E \times E$

$\Longrightarrow$ $f = g$ on $\underline{E \times E \times E}$.

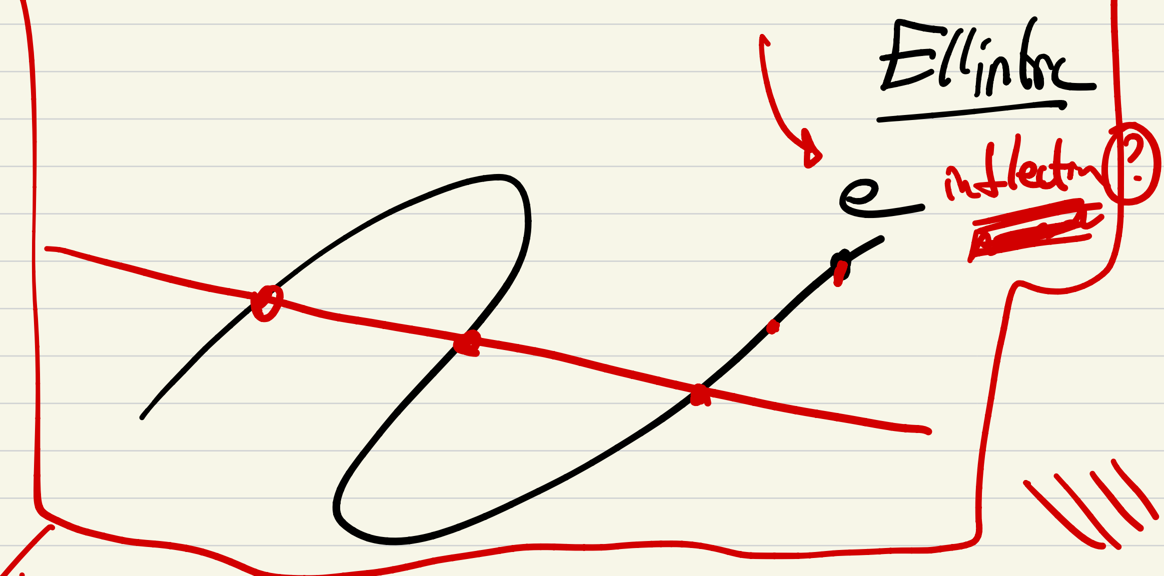
$$\left[E \times E \dashrightarrow E \right]$$

$+$

$$E = X(y^2 - (x^3 - ax - b)) \cup (0:1:0)$$

$(0:1:0) = e$

is an abelian, proper gp



$\boxed{p+q+r=e} \iff$ p, q, r are collinear.

$q=-p \iff$ p, q, e are collinear.