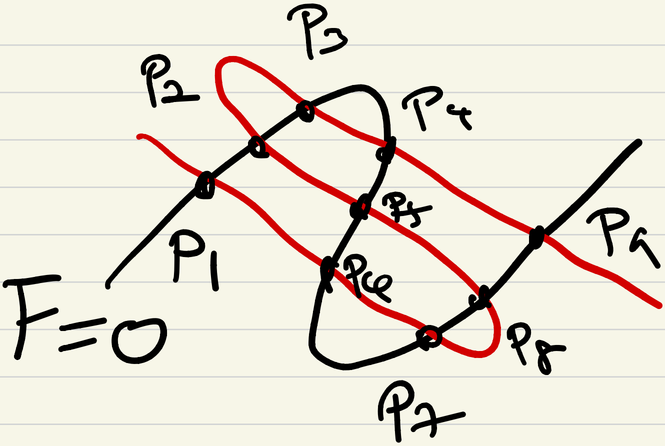
6130-28

Linear series in $\mathbb{P}^2$:



There is something special about the 9 pts. of intersection of two cubic plane curves.

* Any cubic vanishing at $p_1, \ldots, p_8$ is of the form $\underline{\lambda F + \mu G}$. *

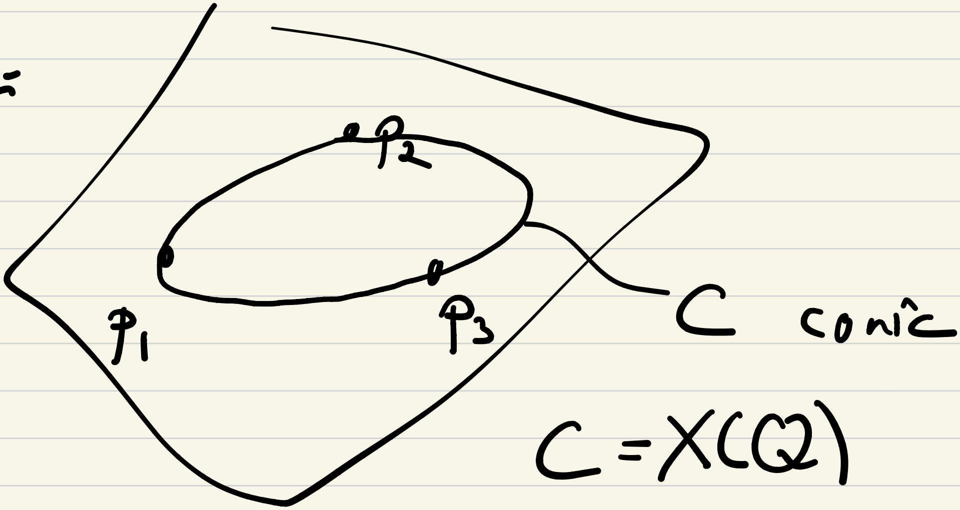
Backsround: (Reid's Notation)

$$S_0 = k[x, yz].$$

$$S_d(p_1, \ldots, p_n) = \left\{ F \in S_d \;\middle|\; \begin{array}{l} F(p_i) = 0 \\ \forall i \end{array} \right\}$$

This is a vector space!

e.g.



$$C = X(Q)$$

$$Q \in S_2(P_1, P_2, P_3).$$

Rmk (1): $\dim_k S_d = \binom{d+2}{2}$

$\dim_k S_1 = 3$

$\dim_k S_2 = 6$

$\dim S_3 = 10$

---

1 or 0

(2) (cod 1)

$$S_d \supset S_d(P_1) \supsetneq S_d(P_1,P_2) \supset \cdots$$

$$S_d(P_1) = \ker(\mathrm{ev}_{P_1} : S_d \to \mathbb{C})$$

Either: $S_d(P_1, \dots, P_{n+1}) = S_d(P_1, \dots, P_n)$

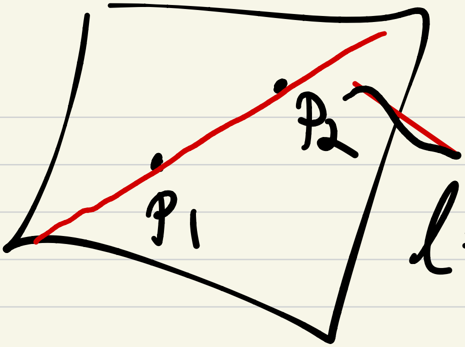
or $S_d(P_1, \dots, P_{n+1}) \overset{\text{cod } 1}{\subseteq} S_d(P_1, \dots, P_n)$

(3) If $p_1, p_2, \ldots, p_n$ are chosen in (successive) open sets, and $n \leq \binom{d+2}{2}$, then

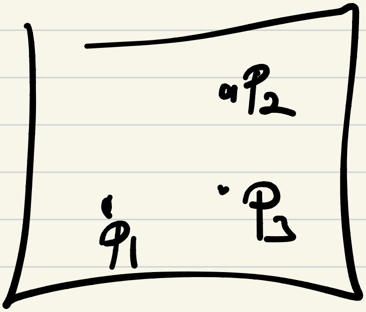
$$\dim_k S_d(p_1, \ldots, p_n) = \binom{d+2}{2} - n.$$

Interesting stuff when $p_1, \ldots, p_n$ aren't "general" in this sense.

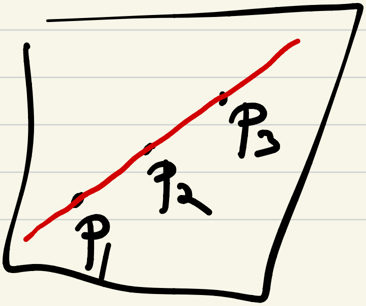
E.g.



$$S_1 \supsetneq S_1(p_1) \supsetneq \underset{\substack{\| \\ k\cdot L}}{S_1(p_1,p_2)}$$

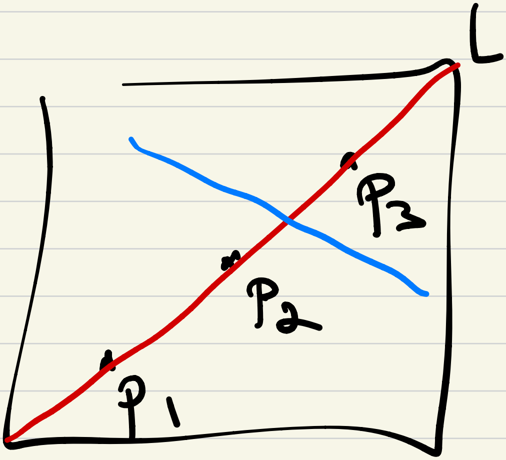


$$S_1(p_1,p_2,p_3)=0$$



$$\overset{\substack{X(L) \\ \|}}{S_1(p_1,p_2,p_3)} = S_1(p_1,p_2)$$

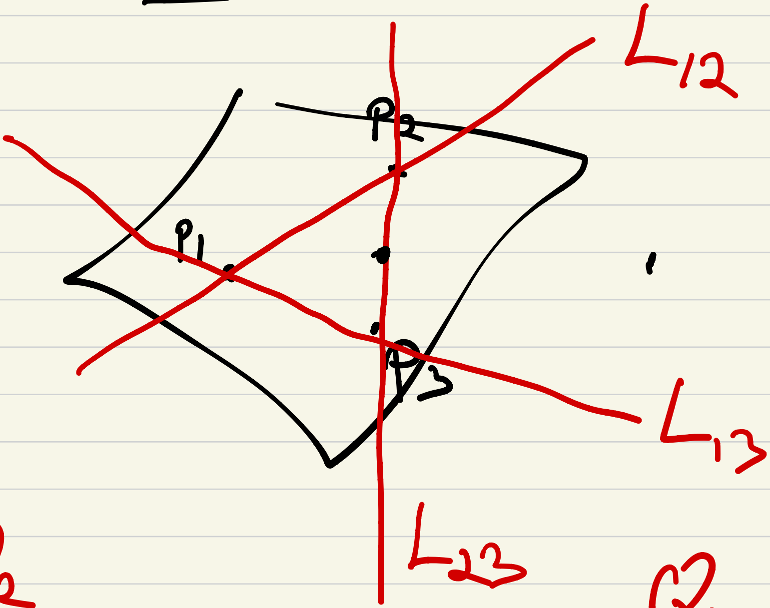
$(d=2)$ $\dim S_2(P_1,P_2,P_3)=6-3$
$=3$



Any conic through collinear pts P_1,P_2,P_3 is divisible by L

For collinear pts, $S_2(P_1,P_2,P_3)=L\cdot S_1$
$(\dim=3)$

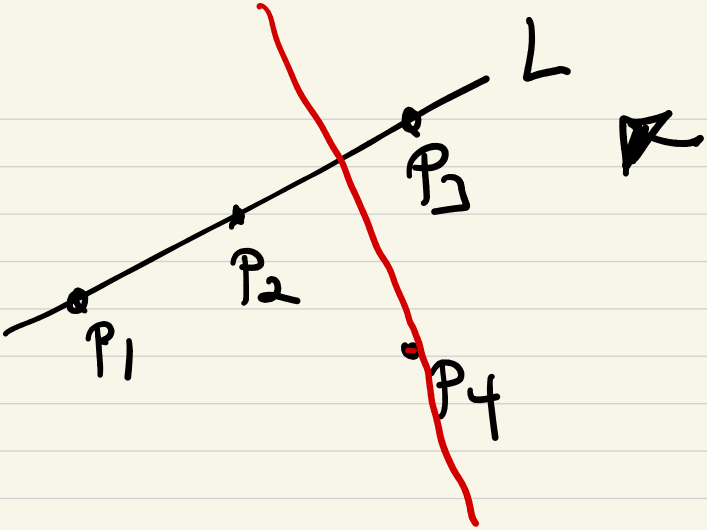
__For__ __non__-collinear __pts__



$\underline{\overset{Q_2}{\overset{\shortparallel}{L_{12}\cdot L_{23}}}}$, $\overset{Q_3}{\overset{\shortparallel}{L_{13}\cdot L_{23}}}$, $\underline{\overset{Q_1}{\overset{\shortparallel}{L_{12}\cdot L_{13}}}}$

are a basis for

$$\underline{\underline{S_2(p_1, p_2, p_3)}} .$$



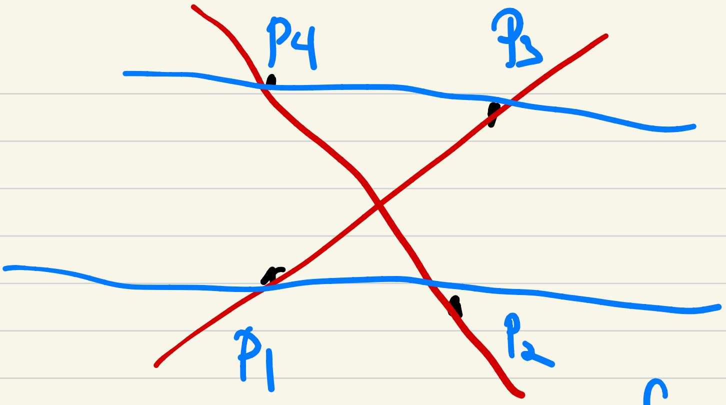
$$\underline{S_2(P_1,P_2,P_3,P_4) = L\cdot S_1(P_4)}$$

<u>Rmk:</u>

$$\dim S_2(P_1,P_2,P_3,P_4) = 3$$

$\Longleftrightarrow$ P_1,P_2,P_3,P_4 are <u>collinear</u>

$$\left(S_2(P_1,P_2,P_3,P_4) = L\cdot S_1 \right)$$

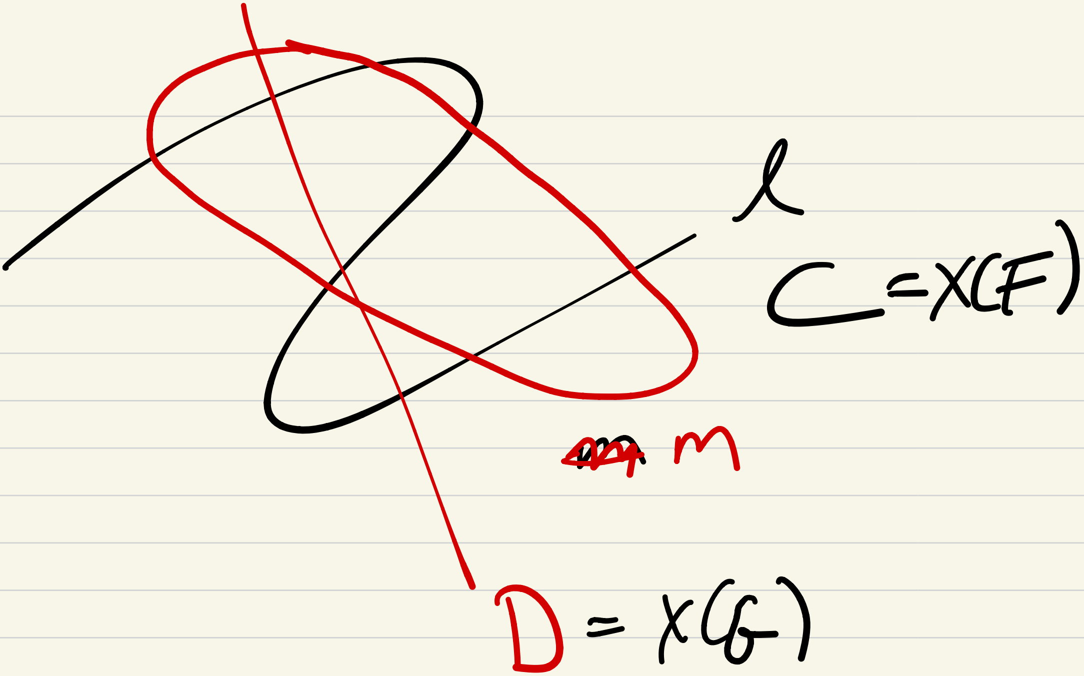


basis!

for $S_2(p_1, p_2, p_3, p_4)$

Suppose $C = X(F)$ is an irreducible curve of degree l and

$D = X(G)$ is (not necessarily irreducible) of degree m.



Proposition: If $\# C \cap D > \ell m$

then $F \mid G$.

i.e. C is a component of D.

(Version of Bézout's Thm)

__Pf__: Consider the Hilbert poly. of $S_\bullet / \langle F \rangle$

$$\begin{matrix} S_\bullet / \langle F \rangle \\ \parallel \\ A_\bullet \end{matrix} \quad (\text{for } C = X(F))$$

__Compute__:

$$S_\bullet(-l) \underset{\cdot F}{\subset} S_\bullet \twoheadrightarrow S_\bullet / F \cdot S_\bullet(-l)$$

$$H_{A_\bullet}(d) = \overset{\swarrow \binom{d+2}{2}}{H_{S_\bullet}(d)} - \overset{\swarrow \binom{d-l+2}{2}}{H_{S_\bullet(-l)}(d)}$$

- $H_{A_\bullet}(d) = ld + \left(1 - \binom{l-1}{2}\right)$

$$\begin{tikzcd} H(d-m) & H(d) & S_\bullet/\langle F\rangle & \\ A_\bullet(-m) \arrow[r, hook, "\cdot G"] & A_\bullet \arrow[u, phantom, "="] \arrow[rr] & & S_\bullet/\langle F,G\rangle \end{tikzcd}$$

$\uparrow$ $\qquad$ (domain)

If G is not divisible by F, then $\cdot G$ is injective!

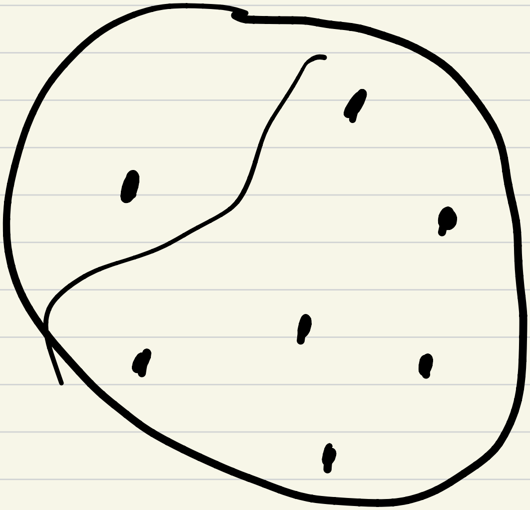
$$\mathrm{Hilb}_{S_\bullet/\langle F,G\rangle}(d) = l \cdot m.$$

$$\| $$

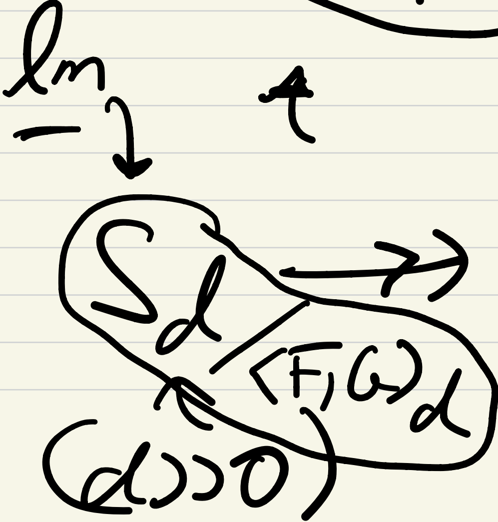
$$(dl + \text{constant}) - ((d-m)l \quad + \text{const})$$

$$= lm$$

"is the homog. coord ring of $X(F,G)$"



$X(F,G)$ $= C \cap D$



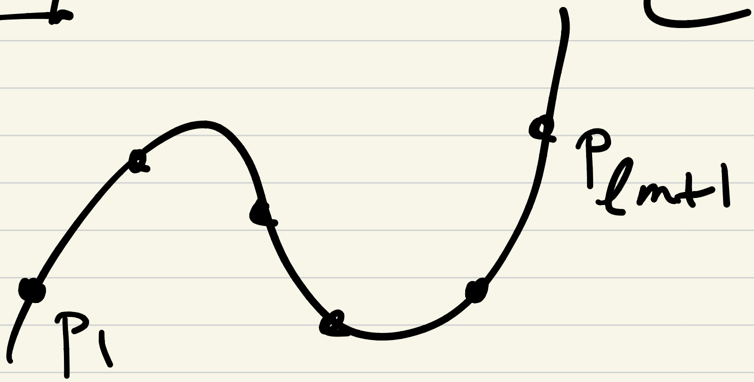
$\longrightarrow \{\underline{\text{all}}\ \underline{\text{fcns.}}\ \text{of}\ |C \cap D|\}$

$\Rightarrow \boxed{\#\ \text{pts} < \underline{lm}}$

So $\#|C\cap D| > lm$

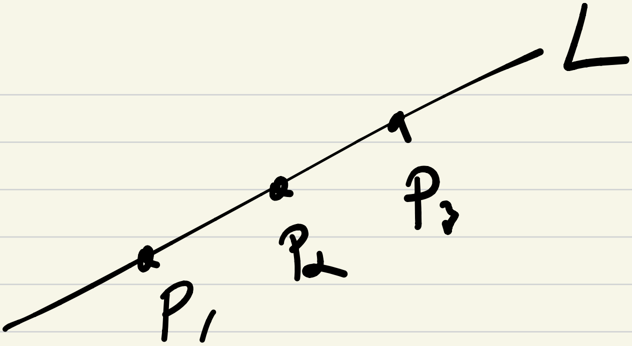
$\Rightarrow G/F$ (!)

Corollary: $\overset{X(F)}{\overset{\shortparallel}{C}}$



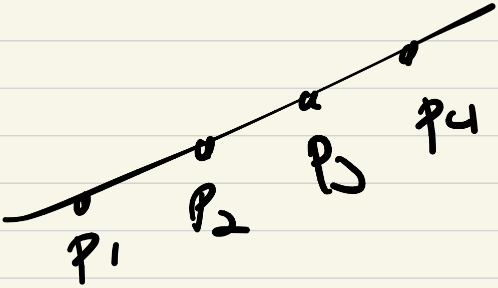
$$S_m\underset{\substack{\rotatebox{90}{$\in$}\\ F\cdot G}}{\underline{(p_1,\dots,p_{lm+1})}} = F\cdot S_{m-l}$$

E.g.

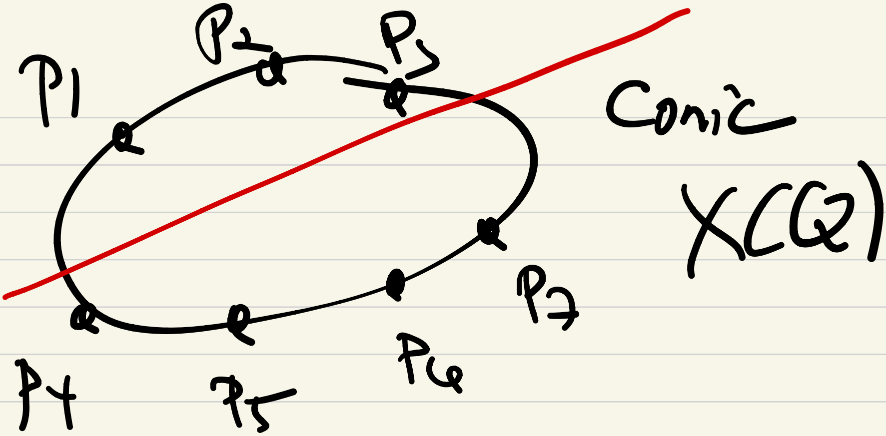


$$S_2(P_1, P_2, P_3) = L - S_1$$

$$(3 > 2 \cdot 1)$$

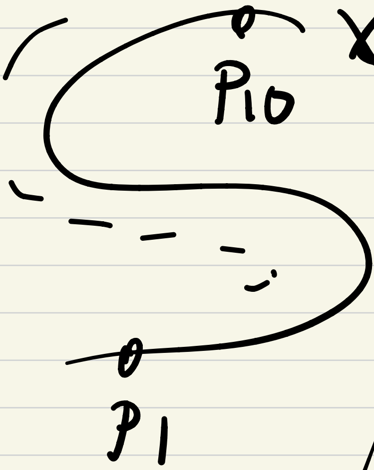


$$\boxed{S_3(P_1, P_2, P_3, P_4) = L - S_2}$$



$$\boxed{S_3(P_1, \ldots, P_7) = Q \cdot S_1}$$

$7 > 3 \cdot 2$



$X = X(F)$

$S_3(P_1, \ldots, P_{10})$

$10 > 3 \cdot 3$

$= \underline{k \cdot F}$