Algebraic Sets

A is a comm. ring w/ 1.

Def: A is Noetherian if every asc. chain of ideals in A eventually stabilizes:

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq A$$

$$\bigcup_{k=1}^{\infty} I_k = I = I_n \text{ for some } n.$$

i.e.

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = I_{n+1} = \cdots$$

Ex. Show that A is Noeth. $\Leftrightarrow$ every ideal in A is finitely generated.

E.g. k is Noetherian

$\mathbb{Z}$, $k[x]$; any PID

$\xleftarrow{\text{divisors}}$

$$\langle n \rangle \subseteq \langle d \rangle \subseteq \ . \ . \ . \ \subseteq \mathbb{Z}$$

$d \mid n$

Lemma: If A is Noetherian and M is a f.g. A-module, then every submodule

$$N \subseteq M \text{ is finitely generated.}$$

Rmk

$$\overset{m_1, \dots, m_k}{M} \overset{g}{\twoheadrightarrow} \overset{\bar{m}_1, \dots, \bar{m}_k}{M/N}$$

is obviously f.g.

Pf: (1) Reduce to <u>free</u> <u>modules</u>.

$$\begin{tikzcd} g^{-1}(N) \arrow[r] \arrow[d, phantom, "\cap"] & N \arrow[d, phantom, "\cap"] \\ A^k \arrow[r, two heads, "g"] & M \end{tikzcd}$$

(2) Prove it for free modules by induction:

$$\begin{tikzcd}
M \arrow[r] & N \arrow[r] & q(N) = I \\
0 \arrow[r] & A^{k-1} \arrow[r] \arrow[u, phantom, "\cup" description] & A^k \arrow[r, "q"] \arrow[u, phantom, "\cup" description] & A \arrow[r] \arrow[u, phantom, "\cup" description] & 0
\end{tikzcd}$$

f.g.   f.g.

$m_1, \ldots, m_\ell$ + lifts of generators
gen. of M $f_1, \ldots, f_j$ of I

= generating set for N.

$\square$

Thm (Basis Thm)

If A is Noeth, then $A[x]$ is Noetherian.

Pf: (Careful accounting!)

Suppose $J \subset A[x]$ is an ideal.

$$\begin{matrix} \psi \\ f = a_d x^d + \text{lower order} \end{matrix}$$

Create an ascending chain:

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq A$$

$$I_d = \left\{ a_d \in A \;\middle|\; \begin{array}{l} \exists f = a_d x^d + \cdots \\ \text{in } J \end{array} \right\}$$

(1) I_d are ideals. ✓

$$I_d \ni a_d + b_d \leftarrow \underset{\in J}{f+g}$$

$a_d \leftarrow f$, $b_d \leftarrow g$

$$I_d \ni a a_d \sim \underset{\in J}{af}$$

(2) $I_d \subseteq I_{d+1}$

$$I_d \ni a_d \leftrightarrow f \qquad f \cdot x \leftrightarrow \underset{I_{d+1}}{a_d}$$

Eg. $J = \langle x \rangle$ $\qquad ax \in J$

$I_0 = \langle 0 \rangle$

$I_1 = A$ $\qquad ax \in J$

$I_2 = A$ $\qquad ax^2 \in J$

$\vdots$

$J = \langle 2, x \rangle \subseteq \mathbb{Z}[x]$

$I_0 = \langle 2 \rangle$

$I_1 = \langle 1 \rangle = \mathbb{Z}$

$\vdots$

$$I_0 \subseteq \dots \subseteq I_d \dots \subseteq A$$

- $I_n = I_{n+1} = \dots$

$+$

- <u>Each</u> $I_0, \dots, I_n$ is f.g.

$\rightsquigarrow$ choose generators for

$I_0, \dots, I_n$

and then choose poly. representing them. These generate J. $\square$

Cor: $\underline{k}[x_1, \dots, x_n]$ are Noetherian.

Ex: Let $X \subset k^n$. Then

$$I(X) = \{f \in k[x_1, \dots, x_n] \mid f|_X \equiv 0\}$$

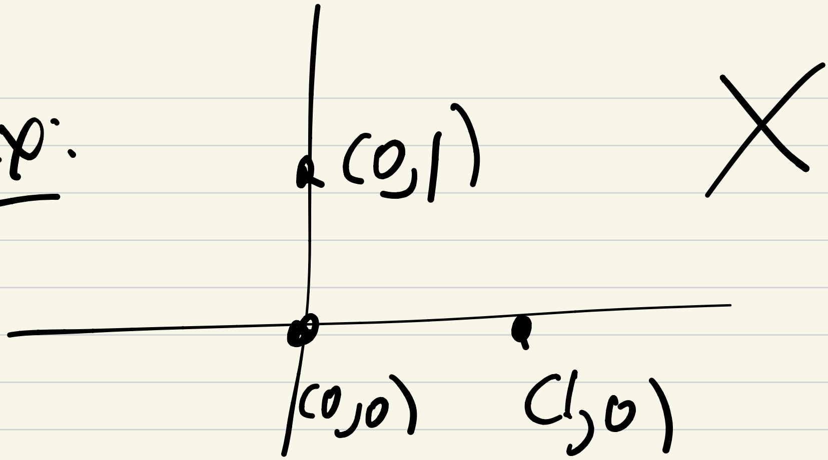
$f + g$

hf

This is an ideal.

Finitely generated!

Ex:



$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ $\qquad J = I(X)$

$I_d \subseteq k[x_1]$ $\qquad \cap$ $k[x_1, x_2]$

$I_0 = \langle x_1(x_1 - 1) \rangle$ $\qquad g(0) = 0$

$I_1 \ni \underline{f(x_1)} \iff$

$\exists g(x_1)$ s.t. $g(x_1) + x_2 f(x_1) \in I(X)$

$X : \{\text{ideals in } k[x_1, \dots, x_n]\}$

$$\xrightarrow{X(I)} \{\text{subsets of } k^n\}$$

$I : \{\text{subsets of } k^n\}$

$$\longrightarrow \{\text{ideals in } k[x_1, \dots, x_n]\}$$

<u>Def</u>: X is alg. if $X = X(I)$

I is geo. if $I = I(X)$

Simple Observations:

- $I \subseteq J \Rightarrow X(I) \supseteq X(J)$
- $X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$
- $X \subseteq \underline{X(I(X))}$ ($= \overline{X}$ in the Zariski top)
- $\underset{\subseteq\, \mathrm{rad}(I)}{I} \subseteq I(X(I))$ ($= \mathrm{rad}(I)$ when $k = \overline{k}$)

Def: In $\overline{A/I}$ A, $\mathrm{rad}(I) = \{a \mid a^n \in I$ for some $n\}$