Math 6130

Fall 2020

MWF 11:50-12:40

Math 6130

Algebraic Geometry I

www.math.utah.edu/~bertram/6130

§0 Preview

§1 Algebraic Sets in k^n

<u>k</u> <u>field</u>

$\mathrm{char}(k) = 0$ ☺

$\mathrm{char}(k) = p > 0$ $\left(\frac{d}{dx} x^p = 0\right)$

$k = \bar{k}$ (need for varieties)

(schemes handle $k \neq \bar{k}$ nicely)

Algebraic Geometry:

$$f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n)$$

$$= 0$$

$$\Downarrow \qquad \subset k^n$$

Loci: $X(f_1, \dots, f_m) = 0$

$$= \{a \in k^n \mid f_1(a) = \dots = f_m(a) = 0\}$$

Would Like:

$$\dim(X(f_1, \dots, f_m)) \geq n - m.$$

Bad Example:

$$X(x^2+y^2) \subset \mathbb{R}^2$$

$$\| \qquad \|$$

$$(0,0)$$

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when $\underline{k=\overline{k}}$, the dim inequality <u>will</u> <u>hold</u>.

<u>E.g.</u> $k=\mathbb{C}$

$k=\overline{\mathbb{Q}}$ $\qquad$ $k=\overline{\mathbb{F}_q}$.

First thing: $k[x_1, \dots, x_n]$

$\vee$

$f_1, \dots, f_m \longleftrightarrow \langle f_1, \dots, f_m \rangle$

$\not\Leftarrow$ $\Rightarrow$

$X(\langle f_1, \dots, f_m \rangle)$ $\quad$ f all ideals

$$f = \sum g_i f_i$$

Ideals: $I \subset k[x_1, \dots, x_n]$

Algebraic sets $X \subset k^n$:

alg. if $X = X(I) \subset k^n$

$X =$

Ex: $\{(t, t^2, t^3) \mid t \in \mathbb{C}\}$

x, y, z $\qquad \subset \mathbb{C}^3$

is an <u>algebraic set</u>:

$x^2 - y \qquad X(x^2 - y) = \{(t, t^2, *)\}$

$x^3 - z \qquad X(x^2 - z) = \{(t, *, t^3)\}$

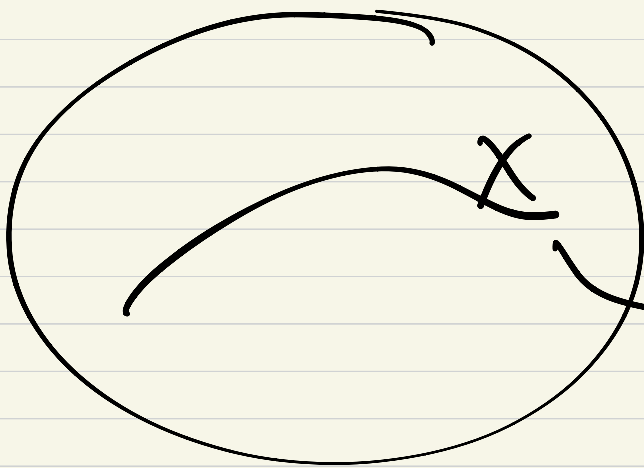
$$X = X(\langle x^2 - y, x^3 - z \rangle)$$

$$\in$$

$$xz - y^2$$

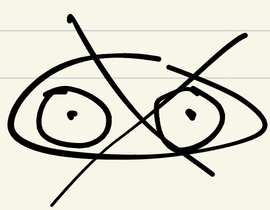
<u>Topology</u>: The alg. sets in k^n are the closed sets of the <u>Zariski</u> <u>Topology</u> on k^n.

k^n $(k = \mathbb{C})$



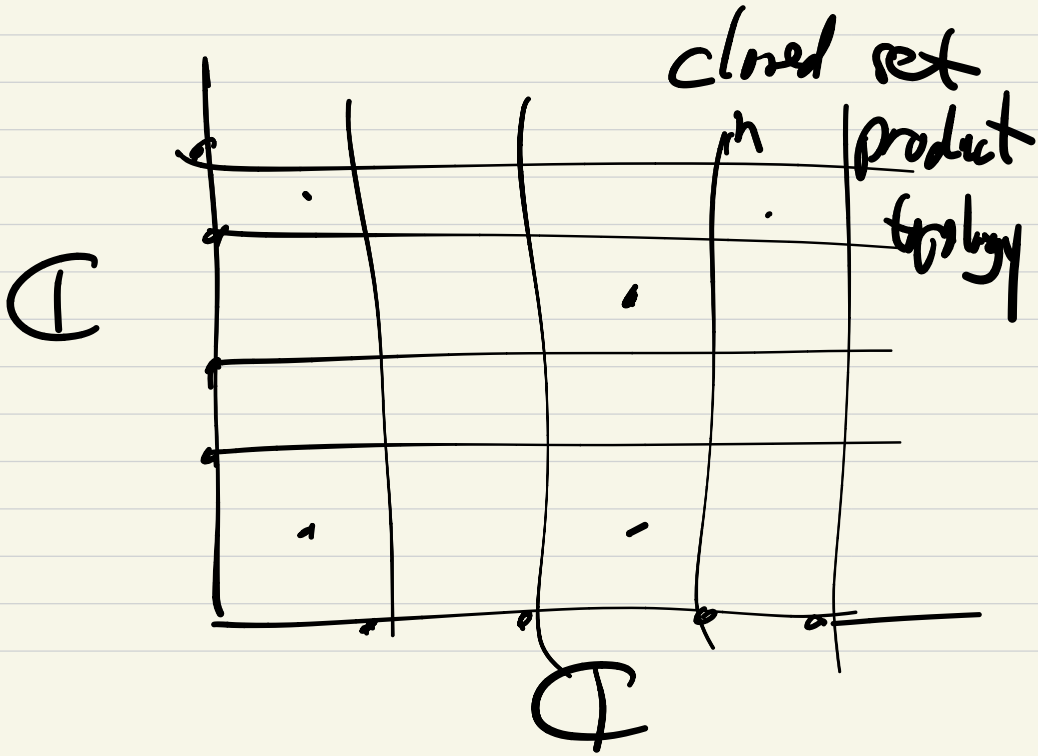
closed sets have measure 0.

open sets always intersect!

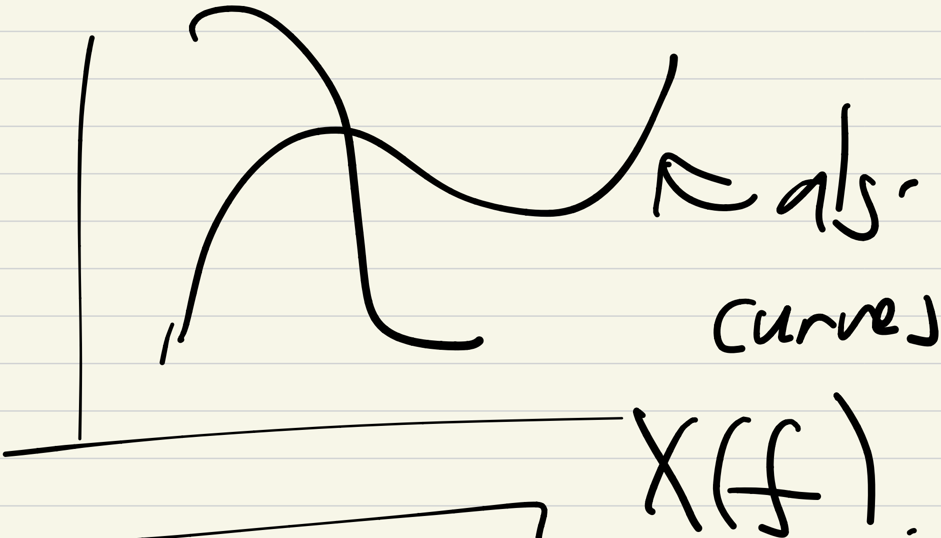


Zariski Topology is separated
(morally Hausdorff)

Problem: Products don't have the product topology!!



Zariski Top. on $\mathbb{C}^2$:



An affine variety:

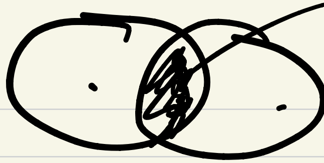
$X(\text{prime ideal}) \subset k^n$

+

induced topology

+

sheaf of regular functions

Geometry:  gluing

Manifold (Hausdorff)

+

Topology / basis of open balls in $\mathbb{R}^n$

+

Sheaf of functions top

e.g. continuous functions

diff geo differentiable functions

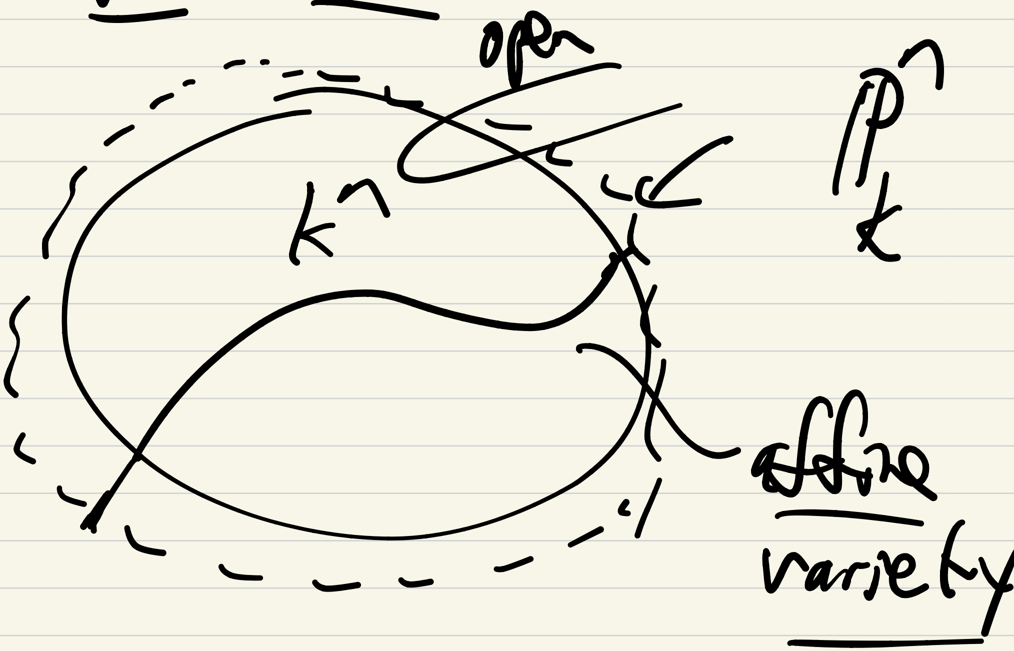
analytic functions

analy. geo

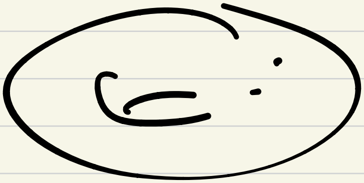
Algebraic Geometry

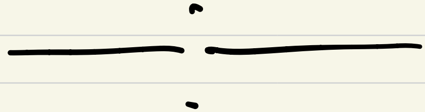
No open balls ☹

Good Open sets = affine varieties



$\mathbb{P}^n_K$ = projective space is proper (= morally compact)





Abstract variety

gluing by rational functions

open affine varieties

$\mathbb{P}^n_k$ proper over Spec k

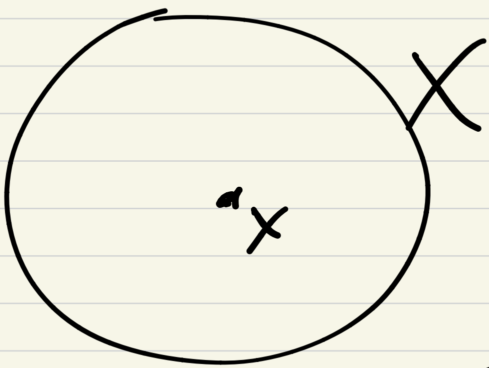
Features of Varieties

Local: Non-singularity

$$x(y^2 - x^3)$$

Local: (Algebraic property)

Normality: (weaker than non-singular)



$$\mathcal{O}_{X,x} \subset k(X)$$

integrally closed

local ring of rational fcns defined at x