Lesson Twelve

Math 6080 (for the Masters Teaching Program), Summer 2020

Euler's Proof of Euclid's Theorem. Recall that the *harmonic series:*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} + \cdots$$

diverges, which is to say that it eventually surpasses every natural number.

On the other hand, the *geometric series* of the powers of $1/2$ converges:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$$

For **every** prime number p, the geometric series of powers of $1/p$ converges:

$$1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n} + \cdots = \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1}$$

Now suppose we multiply two of them:

$$(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} + \cdots)$$

On the one hand:

$$2\left(\frac{3}{3-1}\right) = 3$$

but on the other hand, by distributing the multiplication, we obtain:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{1}{8} + ... + \frac{1}{2^m 3^n} + \cdots = 3$$

which is the sum of the reciprocals of every number with only 2 and 3 as factors.

If we do this for **all** the primes, we get:

$$(*) \text{ harmonic series} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{p}{p-1} \cdots$$

In particular, there cannot be finitely many prime because the left side diverges!

But Euler gets an even better result. Let's review some Calculus.

(i) the sum of the harmonic series to $1/n$ is trapped between $\ln(n)$ and $\ln(n)+1$. We can numerically check this with Python! Thus the harmonic series very slowly diverges, dancing an intimate slow dance with the natural logarithm.

(ii) the Maclaurin power series for $\ln(1-x)$ is:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

In particular, setting $x = -1$, we get:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

which is the *alternating harmonic series*, whose convergence we can again check numerically with Python. This converges fairly quickly. If n is odd, then:

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{n-1} < \ln(2) < 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{n} < \ln(2) + \frac{1}{n}$$

because the series is alternating, so the series summed up to $1/n$ is trapped between $\ln(2)$ and $\ln(2) + 1/n$.

(iii) Now let's take the natural logarithm of both sides of $(*)$ above:

$$\ln(\text{harmonic series}) = \ln\left(\prod_p \left(\frac{1}{1-\frac{1}{p}}\right)\right) = \sum \ln\left(\frac{1}{1-\frac{1}{p}}\right)$$
$$= \sum -\left(-\frac{1}{p} - \frac{1}{2p^2} - \frac{1}{3p^2} - \cdots\right)$$

If we rarrange* the terms of this infinite sum of infinite sums, we get:

$$\ln(\text{harmonic series}) = \sum_p \frac{1}{p} + \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \cdots\cdots$$

and all the terms (and their infinite sum!) other than the first term **converge**.

The consequence of this is:

$$\ln(\text{harmonic series}) < \sum_p \frac{1}{p} + \text{constant}$$

But $\ln(\ln(n))$ goes to infinity as n goes to infinity, so the sum of $1/p$ diverges! As noted earlier, this says more than simply that there are infinitely many primes.

Dirichlet's Theorem is a variation in which one fixes a "modulus" m and asks:

For each remainder r between 0 and $m-1$, what "proportion" of the primes satisfy:

$$p\%m = r$$

For example, suppose $m = 3$. Then:

(0) 3 is divisible by 3.

(1) $7, 13, 19, ...$ satisfy $p\%3 = 1$.

(2) $2, 5, 11, 17, ...$ satisfy $p\%3 = 2$.

As another example, suppose $m = 4$. Then:

(0) Nothing

(1) $5, 13, 17, 29, ...$ satisfy $p\%4 = 1$.

(2) 2 satisfies $p\%4 = 2$. Nothing else.

(3) $3, 7, 11, 19, ...$ satisfy $p\%4 = 3$.

Dirichlet's Theorem. For each fixed modulus m.

(i) If $\gcd(m, r) \neq 1$, then at most one prime satifies $p\%m = r$.

(ii) For all the remainders r that **do** satisfy $\gcd(m, r) = 1$, the numbers of primes between 1 and n satisfying $p\%m = r$ are approximately the same. In an appropriate sense, the infinitely many primes are evenly distributed among these remainders.

Remark. The proof of (i) is easy. If $p\%m = r$, then:

$$\text{gcd(m,r)} = \text{gcd(m,p)} = \text{d}$$

and so d divides p. But if p is prime, then we must have $d = 1$ or $d = p$.

(ii) is hard.

Our Challenge. To write Python code to check (ii) numerically.

The Strategy. Use the Sieve of Eratosthenes to create a list of lists.

```
Dirichlet = []
for r in range(m):
    Dirichlet = Dirichlet + [[ ]]
```

This creates a list of m empty lists, with Dirichlet[r] = [].

Now we feed into each Dirichlet[r] all the primes in the Sieve with p%m = r.

Then we compare the values len(Dirichlet[r]) as r ranges from 0 to $m - 1$ and numerically "see" the even distribution of the primes from Dirichlet's Theorem. .

Exercise. Write the code to do this.