Math 6130 Notes. Fall 2002.

4. Projective Varieties and their Sheaves of Regular Functions. These are the geometric objects associated to the **graded** domains:

$$\mathbf{C}[x_0, x_1, ..., x_n]/P$$

(for homogeneous primes P) defining"global" complex algebraic geometry.

Definition: (a) A subset $V \subseteq \mathbf{CP}^n$ is *algebraic* if there is a homogeneous ideal $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$ for which $V = V(I)$.

(b) A homogeneous ideal $I \subset \mathbf{C}[x_0, x_1, ..., x_n]$ is *radical* if $I = \sqrt{I}$.

Proposition 4.1: (a) Every algebraic set $V \subseteq \mathbf{CP}^n$ is the zero locus:

$$V = \{F_1(x_1, ..., x_n) = F_2(x_1, ..., x_n) = ... = F_m(x_1, ..., x_n) = 0\}$$

of a finite set of *homogeneous* polynomials $F_1, ..., F_m \in \mathbf{C}[x_0, x_1, ..., x_n]$.

(b) The maps $V \mapsto I(V)$ and $I \mapsto V(I) \subset \mathbf{CP}^n$ give a bijection:

$$\{\text{nonempty alg sets } V \subseteq \mathbf{CP}^n\} \leftrightarrow \{\text{homog radical ideals } I \subset \langle x_0, ..., x_n\rangle\}$$

(c) A topology on $\mathbf{CP}^n$, called the *Zariski topology*, results when:

$$U \subseteq \mathbf{CP}^n \text{ is open } \Leftrightarrow Z := \mathbf{CP}^n - U \text{ is an algebraic set (or empty)}$$

Proof: Note the similarity with Proposition 3.1. The proof is the same, except that Exercise 2.5 should be used in place of Corollary 1.4.

Remark: As in §3, the bijection of (b) is "inclusion reversing," i.e.

$$V_1 \subseteq V_2 \Leftrightarrow I(V_1) \supseteq I(V_2)$$

and irreducibility and the features of the Zariski topology are the same.

Example: A *projective hypersurface* is the zero locus:

$$V(F) = \{(a_0 : a_1 : ... : a_n) \in \mathbf{CP}^n \mid F(a_0 : a_1 : ... : a_n) = 0\} \subset \mathbf{CP}^n$$

whose irreducible components, as in §3, are obtained by factoring F, and the *basic open sets* of $\mathbf{CP}^n$ are, as in §3, the complements of hypersurfaces.

Definition: A *projective variety* is an irreducible closed subset $X \subseteq \mathbf{CP}^n$. The *Zariski topology* on X is induced from the Zariski topology on $\mathbf{CP}^n$ and the *hypersurfaces* and *basic open sets* in X are the (proper) intersections of X with hypersurfaces in $\mathbf{CP}^n$ and their complements in X, respectively. The *homogeneous coordinate ring* is the domain $\mathbf{C}[X] := \mathbf{C}[x_0, x_1, ..., x_n]/P$ (which is now graded!) where P is the homogeneous prime ideal $I(X)$.

Observation: Nonempty closed subsets $Z \subseteq X$ correspond to homogeneous radical ideals $I \subset \langle \overline{x}_0, ..., \overline{x}_n \rangle \subset \mathbf{C}[X]$, and irreducible closed subsets in X correspond to homogeneous prime ideals $P \subset \langle \overline{x}_0, ..., \overline{x}_n \rangle \subset \mathbf{C}[X]$.

But now what should a regular function on a projective variety X be? Constant functions we understand, but the non-constant elements of $\mathbf{C}[X]$ are not functions on X because the value:

$$\overline{F}(a_0 : ... : a_n)$$

always depends upon the choice of a representative of the class $(a_0 : ... : a_n)$ (but we do know whether or not $\overline{F}(a_0 : ... : a_n) = 0$ when $\overline{F}$ is homogeneous).

Definition: A *rational function* on X is an element "of degree zero" in the field of fractions of $\mathbf{C}[X]$. That is, ϕ is a rational function if $\phi = \frac{\overline{F}}{\overline{G}}$ and $\overline{F}, \overline{G} \in \mathbf{C}[X]$ are homogeneous of the same degree. If $\overline{G}(a_0 : ... : a_n) \neq 0$ for some such ratio representing ϕ then $\phi(a_0 : ... : a_n) \in \mathbf{C}$ is well-defined, and we say that ϕ is *regular* at $(a_0 : ... : a_n)$

Observation: The rational functions on X form a **field**, denoted $\mathbf{C}(X)$.

Examples: (a) The field of rational functions on $\mathbf{CP}^n$ itself is:

$$\mathbf{C}\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right) = ... = \mathbf{C}\left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right) = ... = \mathbf{C}\left(\frac{x_0}{x_n}, ..., \frac{x_{n-1}}{x_n}\right)$$
$$\subset \mathbf{C}(x_0, ..., x_n)$$

(b) Let $F(x_0, ..., x_n) = x_0 x_1 - G(x_2, ..., x_n)$ be a quadratic homogeneous (irreducible) polynomial, and let $X = V(F) \subset \mathbf{CP}^n$. Then:

$$\mathbf{C}(X) = \mathbf{C}\left(\frac{\overline{x}_2}{\overline{x}_0}, ..., \frac{\overline{x}_n}{\overline{x}_0}\right) = \mathbf{C}\left(\frac{\overline{x}_2}{\overline{x}_1}, ..., \frac{\overline{x}_n}{\overline{x}_1}\right)$$

Clearly those fields are subfields of $\mathbf{C}(X)$, and if $\frac{\overline{A}}{\overline{B}} \in \mathbf{C}(X)$, then we can remove all occurrences of $\overline{x}_1$ (resp of $\overline{x}_0$) by multiplying top and bottom by a sufficiently high power of $\overline{x}_0$ (resp. $\overline{x}_1$) and using the equation.

This, of course, begs an interesting question:

Question: Are such projective hypersurfaces "isomorphic" to $\mathbf{CP}^{n-1}$?

We will answer this in the affirmative in case $n = 2$ and in the negative in the other cases. We will see, however, in §7 that these quadric hypersurfaces always have an open (dense) subset which is isomorphic to an open (dense) subset of $\mathbf{CP}^{n-1}$. We need for now to figure out what "isomorphic" means.

Definition: The *sheaf* $\mathcal{O}_X$ *of regular functions* on a projective variety is:

$$\mathcal{O}_X(U) = \{\phi \in \mathbf{C}(X) \mid \phi \text{ is regular at all points of } U\} \subset \mathbf{C}(X)$$

This is the same definition as in §3, $\mathcal{O}_X$ is again a sheaf of commutative rings with 1, and if a rational function is zero (as a function) on an open set, then again it is zero in $\mathbf{C}(X)$ (same proof as Proposition 3.4). But recall that in the affine case there were *lots* of regular functions (ie. all of $\mathbf{C}[X]$). In the projective case, by contrast, there are very few:

Proposition 4.2: The only rational functions that are regular at all points of a projective variety X are the constants (i.e. $\mathcal{O}_X(X) = \mathbf{C} \subseteq \mathbf{C}(X)$).

Proof: If the projective Nullstellensatz were the ordinary Nullstellensatz, this would be easy. Given $\phi \in \mathbf{C}(X)$, the homogeneous denominators in the expressions $\phi = \frac{\overline{F}}{\overline{G}}$ generate an ideal I_ϕ, and then any (homogeneous) element of I_ϕ appears as a denominator, as in Proposition 3.3. By assumption $V(I_\phi) = \emptyset$, and the ordinary Nullstellensatz would imply that 1 appears as a denominator, and then $\phi = \frac{a}{1}$ is a constant. But this requires more argument, since applying the projective Nullstellensatz (Corollary 2.1) only gives us:

$$\phi = \frac{\overline{F}_0}{\overline{x}_0^N} = \cdots = \frac{\overline{F}_n}{\overline{x}_n^N}$$

for some N and homogeneous polynomials $\overline{F}_i$ of degree N. We can conclude from this that if $d \geq (n+1)(N-1)+1$ and if $\overline{G} \in \mathbf{C}[X]_d$ is arbitrary, then

$$\overline{G}\phi \in \mathbf{C}[X]_d$$

(because each monomial in a representative of $\overline{G}$ must have some x_i^N in it!) Now iterate this to conclude that $\overline{G}\phi^m \in \mathbf{C}[X]_d$ for all m, and thus:

$$\mathbf{C}[X] \subseteq \mathbf{C}[X] + \phi\mathbf{C}[X] \subseteq \mathbf{C}[X] + \phi\mathbf{C}[X] + \phi^2\mathbf{C}[X] \subseteq \cdots \subseteq \overline{G}^{-1}\mathbf{C}[X]$$

is a sequence of submodules of the finitely generated (in fact, principal!) $\mathbf{C}[X]$-module $\overline{G}^{-1}\mathbf{C}[X]$, which eventually stabilizes since $\mathbf{C}[X]$ is Noetherian.

Thus for some m, we have:

$$\phi^m = \overline{f}_{m-1}\phi^{m-1} + \overline{f}_{m-2}\phi^{m-2} + ... + \overline{f}_0$$

for $\overline{f}_0, ..., \overline{f}_{m-1} \in \mathbf{C}[X]$. But this identity must also hold when we replace the $\overline{f}_i$ by their degree 0 components, and then ϕ satisfies a (monic) polynomial equation with constant coefficients. Since $\mathbf{C}$ is algebraically closed, it follows that the polynomial factors, so that $\prod_{i=1}^m (\phi - r_i) = 0$ for some $r_i \in \mathbf{C}$. But if $\phi = \frac{\overline{F}}{\overline{G}}$, then $\phi(a_0 : ... : a_n) = r_i \Leftrightarrow \overline{F}(a_0 : ... : a_n) = r_i\overline{G}(a_0 : ... : a_n)$, so the locus where $\phi \equiv r_i$ is closed. Since X is irreducible, it follows that one of these loci must be all of X, and then $\phi = r_i \in \mathbf{C}(X)$ for that $r_i \in \mathbf{C}$.

Remark: Recall that the only holomorphic functions on a compact complex manifold are the constants, since on the one hand a continuous function on a compact manifold must have a (global) maximum, and on the other hand, the absolute value of any holomorphic function is "subharmonic" meaning that it can only have a local maximum if it is locally constant! We will see in §6 that projective varieties are, in a very precise sense, analogues in algebraic geometry of compact manifolds.

Definition: An open subset $W \subseteq X$ of a projective variety with sheaf:

$$\mathcal{O}_W(U) := \mathcal{O}_X(U)$$

is a *quasi-projective variety* (which we will shorten to *variety*). A *regular map* is, as always, a continuous map that pulls back regular functions.

Corollary 4.3: The only regular maps $\Phi : X \to Y$ from a projective variety X to an affine variety Y are the constant maps.

Proof: Suppose $Y \subset \mathbf{C}^n$ has coordinate ring $\mathbf{C}[y_1, ..., y_n]/P$. Then the regular functions $\overline{y}_1, ..., \overline{y}_n \in \mathcal{O}_Y(Y)$ must pull back to regular functions on X, which are constant by Proposition 4.2. If we let $\Phi^*(\overline{y}_i) = b_i \in \mathbf{C}$, then

$$\Phi(a_0 : ... : a_n) = (b_1, ..., b_n) \ \ \text{(independent of } (a_0 : ... : a_n) \in X)$$

since the ith coordinate is $\overline{y}_i(\Phi(a_0 : ... : a_n)) = \Phi^*(\overline{y}_i)\,((a_0 : ... : a_n)) = b_i$.

On the other hand, there are plenty of interesting regular maps from one projective variety to another projective variety. The situation is similar to but more subtle than Proposition 3.5 for affine varieties.

Definition: If $X \subseteq \mathbf{CP}^m$ and $Y \subseteq \mathbf{CP}^m$ are projective (or quasi-projective) varieties, then a mapping from an open (dense) domain $U \subseteq X$:

$$\Phi : X --> Y$$

(the broken arrow indicates that U may not be all of X) is a *rational map* if, for each $(a_0 : ... : a_n) \in U$, there is a degree d and homogeneous polynomials $F_0, ..., F_m \in \mathbf{C}[x_0, x_1, ..., x_n]_d$ such that not every $F_i(a_0 : ... : a_n) = 0$ and:

$$\Phi(b_0 : ... : b_n) = (F_0(b_0 : ... : b_n) : ... : F_m(b_0 : ... : b_n))$$

for all $(b_0 : ... : b_n)$ in the neighborhood $X - \cap_{i=0}^m V(\overline{F}_i) \subseteq U$ of $(a_0 : ... : a_n)$.

Remark: If $b = (b_0 : ... : b_n) \in \mathbf{CP}^n$, then $(F_0(b) : ... : F_m(b)) \in \mathbf{CP}^m$ is well-defined for $b \in \mathbf{CP}^n$, provided that the F_i are all homogeneous of the same degree and not all of the $F_i(b)$ are zero, since in that case:

$$(F_0(\lambda b) : ... : F_m(\lambda b)) = (\lambda^d F_0(b) : ... : \lambda^d F_m(b)) = (F_0(b) : ... : F_m(b))$$

The subtlety here lies in the fact that the choice of $F_0, ..., F_m$ may depend upon the point $(a_0 : ... : a_n)$, so that the domain U of Φ is usually larger than $X - \cap_{i=0}^m(\overline{F}_i)$ for any single set of $\overline{F}_0, ..., \overline{F}_m$.

Example (The Projective Conic) The conic $C := V(y^2 - xz) \subset \mathbf{CP}^2$ is isomorphic to $\mathbf{CP}^1$, in the sense that there are rational maps:

$$\Phi : C --> \mathbf{CP}^1 \text{ and } \Psi : \mathbf{CP}^1 --> C$$

that are defined everywhere and are two-sided inverses of each other.

The map Ψ is easy to construct:

$$\Psi(a : b) = (a^2 : ab : b^2)$$

which is defined on all of $\mathbf{CP}^1$ and maps to C. The map Φ is in two parts:

$$\Phi(a : b : c) = \begin{cases} (a : b) \text{ on } C - (V(\overline{x}) \cap V(\overline{y})) \\ \\ (b : c) \text{ on } C - (V(\overline{y}) \cap V(\overline{z})) \end{cases}$$

and only taken together do we see that C is the domain since the point $(0 : 0 : 1) \in C$ missed by the first definition is covered by the second, and $(a : b) = (b : c)$ whenever both definitions apply, as a consequence of $y^2 = xz$. Finally, once easily checks that Φ and Ψ are inverses of each other.

Here is the (quasi-)projective version of Proposition 3.5:

Proposition 4.4: If $U \subseteq \overline{U} \subset \mathbf{CP}^n$ is a quasi-projective variety, then the regular maps $\Phi : U \to W \subset \overline{W} \subset \mathbf{CP}^m$ to another quasi-projective variety are the rational maps $\Phi : \overline{U} --> \mathbf{CP}^m$ with U in the domain and $\Phi(U) \subseteq W$.

Proof: Suppose Φ is a rational map, given locally by $\Phi = (F_0 : ... : F_m)$. Then Φ is continuous on open sets in $U - \cap_{i=0}^m V(\overline{F}_i)$ since $\Phi^{-1}(V(G)) = V(G(\overline{F}_0, ..., \overline{F}_m))$, and pulls back regular functions at $b \in W$ to regular functions at each preimage of b, via

$$\Phi^* \left(\frac{\overline{G}}{\overline{H}} \right) = \frac{\overline{G}(\overline{F}_0, ..., \overline{F}_m)}{\overline{H}(\overline{F}_0, ..., \overline{F}_m)}$$

(compare with Proposition 3.5). Thus Φ is a regular map, by definition!

Conversely, given a regular map $\Phi : U \to W$ and $(a_0 : ... : a_n) \in U$ with $\Phi(a_0 : ... : a_n) = (b_0 : ... : b_m) \in W$, then some $b_i \neq 0$ and the rational functions $\frac{\overline{y}_j}{\overline{y}_i} \in \mathbf{C}(\overline{W})$ pull back to $\frac{\overline{F}_j}{\overline{G}_j} \in \mathbf{C}(X)$ with each $\overline{G}_j(a_0 : ... : a_n) \neq 0$. Φ can then be written:

$$\Phi = (F_0 \prod_{j \neq 0} G_j : ... : F_{i-1} \prod_{j \neq i-1} G_j : \prod_j G_j : F_{i+1} \prod_{j \neq i+1} G_j : ... : F_m \prod_{j \neq m} G_j)$$

valid in a neighborhood of $(a_0 : ... : a_n)$, finishing the proof!

Remark: The sheaf definition of a regular map is superior to the rational map definition because it is intrinsic to the variety, and doesn't depend upon the way U "sits" in $\mathbf{CP}^n$ (though it is easier to work with rational maps). It can be tricky to find the precise domain of a rational map in general, but when X is projective space, we do have the following:

Proposition 4.5: If $\Phi : \mathbf{CP}^n --> \mathbf{CP}^m$ is a rational map and

$$\Phi = (F_0 : ... : F_m) \text{ on the open set } U = \mathbf{CP}^n - \cap_{i=0}^m V(F_i)$$

and if the F_i have no common factor, then the domain of Φ is exactly U.

Proof: Given another expression $\Phi = (G_0 : ... : G_m)$, then some $G_i \neq 0$, and $\frac{F_j}{F_i} = \frac{G_j}{G_i}$ because both are equal to $\Phi^*(\frac{y_j}{y_i})$. Thus $G_j F_i = F_j G_i$, and then since the F_i have no common factor, every $G_j = HF_j$ for some fixed H. But then $\cap_{i=0}^n V(G_i) = \cap_{i=0}^n V(F_i) \cup V(H)$ and the Proposition is proved.

Corollary 4.6: If $m < n$, then the only regular maps:

$$\Phi : \mathbf{CP}^n \to \mathbf{CP}^m$$

are the constant maps.

Proof: By Proposition 4.5, such a map would be given by:

$$\Phi = (F_0 : ... : F_m)$$

for **one** choice of $F_0, ..., F_m$, but by Exercise 2.3, we know that $\cap_{i=0}^m V(F_i) \neq \emptyset$ when $m < n$ and $\deg(F_i) > 0$, so such a rational map cannot be regular!

We've discussed maps from projective varieties to affine varieties and to other projective varieties. This leaves us with maps from affine varieties to projective varieties, which are the most fundamental, as they allow us to regard a projective variety as a "manifold" which is "locally affine:"

Proposition 4.7: A projective variety is always covered by affine varieties. Precisely, for a projective variety $X \subseteq \mathbf{CP}^n$, the particular basic open sets:

$$U_i := X - V(x_i)$$

are either empty or isomorphic to affine varieties, and $X = \bigcup_{i=0}^n U_i$.

Conversely, if $Y \subseteq \mathbf{C}^n$ is an affine variety, then the closure $X = \overline{Y} \subseteq \mathbf{CP}^n$ (adding the points at infinity) is a projective variety, for the inclusion:

$$Y \subseteq \mathbf{C}^n \subset \mathbf{CP}^n; \quad (a_1, ..., a_n) \mapsto (1 : a_1 : ... : a_n)$$

and then $Y = U_0$, so in particular every affine variety is quasi-projective.

Proof: For the first part, by symmetry it suffices to assume $i = 0$. If $X = V(P) \subseteq \mathbf{CP}^n$ for a homogeneous prime $P \subseteq \mathbf{C}[x_0, x_1, ..., x_n]$, we define $Y := V(d_0(P)) \subseteq \mathbf{C}^n$ for the dehomogenized ideal $d_0(P)$ (see §2). Then:

$$U_0 = \{(a_0 : a_1 : ... : a_n) \in X \mid a_0 \neq 0\} \subset \mathbf{CP}^n$$

and the bijection: $\Phi : Y \to U_0 \subset X; \quad (a_1, ..., a_n) \mapsto (1 : a_1 : ... : a_n)$ is an isomorphism, with inverse $(a_0 : ... : a_n) \mapsto (\frac{a_1}{a_0}, ..., \frac{a_n}{a_0})$, as you may check. And since $\cup_{i=0}^n U_i = X - \cap_{i=0}^n V(\overline{x}_i)$ and $\cap_{i=0}^n V(\overline{x}_i) = 0$, we get the first part.

For the second part, we only need to know that:

$$\overline{Y} \cap \mathbf{C}^n = Y$$

But if $P = I(Y) \subset \mathbf{C}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}]$ is the ideal of Y, then the homogenized ideal $h(P)$ satisfies $d_0(h(P)) = P$, so $V(h(P)) \subset \mathbf{CP}^n$ is a closed set containing Y, whose intersection with $\mathbf{C}^n$ is Y, and it follows that $\overline{Y} \cap \mathbf{C}^n = Y$, as desired.

Example (Projectivizing an Affine Hypersurface): If we are given:

$$V(f) \subset \mathbf{C}^n \text{ for } f \in \mathbf{C}[x_1, ..., x_n] \text{ of degree } d$$

then as in §2, we should reinterpret:

$$f = f\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right) \in \mathbf{C}\left[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right]$$

and then homogenize to get:

$$\overline{V(f)} = V(F) \subset \mathbf{CP}^n \text{ for } F(x_0, ..., x_n) = x_0^d f\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right)$$

which is the set-theoretic union:

$$V(f) \cup V(F(0, x_1, ..., x_n)) \subset \mathbf{C}^n \cup \mathbf{CP}^{n-1} = \mathbf{CP}^n$$

of the original affine hypersurface and a "hypersurface at ∞."

We can then consider the cover of $\overline{V(F)}$ by open affine hypersurfaces:

$$V(F) = V(f) \cup \bigcup_{i=1}^{n} V(f_i) \text{ for } f_i = F\left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right)$$

For example, consider the affine elliptic curve (for any $\lambda \neq 0, 1$):

$$E_0 := V(y^2 - x(x-1)(x-\lambda)) \subset \mathbf{C}^2$$

If we think of $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$, we get the closure:

$$E = V(x_2^2 x_0 - x_1(x_1 - x_0)(x_1 - \lambda x_0)) = E_0 \cup (0:1:0) \subset \mathbf{CP}^2$$

and then E is covered by E_0 and

$E_2 = E - \{(1{:}0{:}0), (1:1:0), (1:\lambda:0)\} =$

$$V\left(\frac{x_0}{x_2} - \frac{x_1}{x_2}(\frac{x_1}{x_2} - \frac{x_0}{x_2})(\frac{x_1}{x_2} - \lambda\frac{x_0}{x_2})\right)$$

Exercises 4.

1. (a) Describe the Zariski topology on $\mathbf{CP}^1$.

(b) Describe the Zariski topology on the elliptic curve E.

(c) What are the irreducible closed sets for the Zariski topology on $\mathbf{CP}^2$?

2. (a) Prove that the regular map from $\mathbf{CP}^1$ to the rational normal curve:

$$(s:t) \mapsto (s^d : s^{d-1}t : \ldots : t^{d-1}s : t^d)$$

(see Exercise 2.2) is an isomorphism (generalizing the conic example).

(b) Prove that the regular map $\Phi : \mathbf{CP}^1 \to \mathbf{CP}^2$:

$$(s:t) \mapsto (s^3 : st^2 : t^3)$$

is a homeomorphism to $X = V(x_1^3 - x_0x_2)$ but not an isomorphism to X.

3. Consider $m+1$ linear forms $y_0, \ldots, y_m \in \mathbf{C}[x_0, x_1, \ldots, x_n]_1$.

(a) If $m \le n$ and the y_i are linearly independent, describe the domain of the rational map:

$$\Phi : \mathbf{CP}^n \dashrightarrow \mathbf{CP}^m; \ \Phi(a) = (y_0(a) : \ldots : y_m(a))$$

and for each $b \in \mathbf{CP}^m$, describe $\Phi^{-1}(b)$ (in the domain).

(b) If $m \ge n$ and the forms span $\mathbf{C}[x_0, x_1, \ldots, x_n]_1$, describe the image of Φ and prove that Φ is an isomorphism from $\mathbf{CP}^n$ to its image in $\mathbf{CP}^m$.

(c) If $\Phi : \mathbf{CP}^n \to \mathbf{CP}^n$; $\Phi = (F_0 : \ldots : F_n)$ is any regular map such that the F_i have no common factors, then show each *pair* F_i, F_j is relatively prime, and even more, that if $G \in \mathbf{C}[x_0, \ldots, x_n]_e$ is any polynomial that is not divisible by x_i, then $\gcd(F_i, G(F_0, \ldots, F_n)) = 1$. Conclude that if $d > 1$, then each:

$$\mathbf{C}\left(\frac{F_1}{F_i}, \ldots, \frac{F_n}{F_i}\right) \subset \mathbf{C}\left(\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

is a non-trivial field extension, hence that the only automorphisms of $\mathbf{CP}^n$ (i.e. isomorphisms from $\mathbf{CP}^n$ to itself) are given by $n+1$ linear forms.

(d) Find an isomorphism of groups:

$$\mathrm{Aut}(\mathbf{CP}^n) \cong \mathrm{PGL}(n+1, \mathbf{C})$$

between the group $\mathrm{Aut}(\mathbf{CP}^n)$ of regular automorphisms of $\mathbf{CP}^n$ and the group $\mathrm{PGL}(n+1, \mathbf{C})$ of invertible $n+1 \times n+1$ matrices modulo scalars.

4. (a) If $\Phi : \mathbf{CP}^n -\!\!-> \mathbf{CP}^m$ is one of the rational maps of Exercise 3(a), prove that there is an automorphism $\alpha \in \mathrm{Aut}(\mathbf{CP}^n)$ such that:

$$(\Phi \circ \alpha)(x_0 : .... : x_n) = (x_0 : ... : x_m)$$

These maps are called the *standard projections*.

(b) If $\Phi : \mathbf{CP}^n \to \mathbf{CP}^m$ is one of the regular maps from Exercise 3(b), prove that there are automorphisms $\alpha \in \mathrm{Aut}(\mathbf{CP}^n)$ and $\beta \in \mathrm{Aut}(\mathbf{CP}^m)$ such that:

$$(\beta \circ \Phi \circ \alpha) = (x_0 : ... : x_n : 0 : ... : 0)$$

(c) Prove that if $y \in \mathbf{C}[x_0, x_1, ..., x_n]_1$ is any nonzero linear form and $X \subset \mathbf{CP}^n$ is a projective variety, then the basic open subset $U = X - V(\overline{y})$ is either empty or else isomorphic to an open dense affine variety.

5. (a) Prove the analogue of Exercise 3.5 (b,c) for quasi-projective varieties.

(b) Prove that $\mathbf{CP}^n$ is not isomorphic to any locally closed subset of $\mathbf{CP}^n$.

6. When $\alpha \in GL(n+1, \mathbf{C})$ acts on the space of linear forms $\mathbf{C}[x_0, x_1, ..., x_n]_1$, then α also acts on each space $\mathbf{C}[x_0, x_1, ..., x_n]_d$ by:

$$\alpha(x_0^{i_1} ... x_n^{i_n}) = \alpha(x_0)^{i_1} ... \alpha(x_n)^{i_n}$$

(a) Prove that for any α and F, the hypersurfaces $V(F)$ and $V(\alpha(F))$ are isomorphic. (We say that $V(F)$ is *projectively equivalent* to $V(\alpha(F))$.)

A *quadratic* homogeneous polynomial $F \in \mathbf{C}[x_0, x_1, ..., x_n]_2$ can be written in a symmetric matrix form as:

$$(a_{ij}) \leftrightarrow \sum_i a_{ii} x_i^2 + \sum_{i<j} a_{ij} x_i x_j$$

(b) Prove the action of α translates to conjugation in this case:

$$\alpha^T (a_{ij}) \alpha \leftrightarrow \alpha(\sum_i a_{ii} x_i^2 + \sum_{i<j} a_{ij} x_i x_j)$$

and then prove that every *quadric* hypersurface $V(F) \subset \mathbf{CP}^n$ is projectively equivalent (hence isomorphic) to exactly one of the following:

$$V(x_0 x_1 + x_2^2 + ... + x_i^2) \text{ for } i \geq 2$$

(so $\mathbf{C}(X) \cong \mathbf{C}(\mathbf{CP}^{n-1})$ for *all* quadric hypersurfaces).

Definition: If $i = n$ above, then quadric $V(F) \subset \mathbf{CP}^n$ is *non-degenerate*.