Math 6130 Notes. Fall 2002.

6. Hausdorffness and Compactness. We would like to be able to say that all quasi-projective varieties are Hausdorff and that projective varieties are the only compact varieties. After all, this is the case when the varieties are manifolds and we consider them with their "ordinary" manifold topologies.

But the Zariski topology is a strange topology, and when we take the literal definitions of Hausdorff and compact, we find the exact opposite!

• All the open subsets $\emptyset \neq U \subseteq Y$ of a quasi-projective variety are dense (as in Proposition 2.3(b)) so a variety (other than a point) is never Hausdorff.

• All open covers $Y = \cup U_i$ admit finite subcovers, since the complements $Y - U_1 \supseteq Y - (U_1 \cup U_2) \supseteq ...$ are a descending chain of closed sets, which is eventually stationary (Proposition 2.3(d)). So all varieties are compact.

This is a puzzle, until we realize that the ordinary definitions rely on the fact that the product of topological spaces has the product topology. This is not the case with the Zariski topology on a product of varieties, and explains why the correct categorical definitions of Hausdorff and compact differ from the naive ones. With the correct definitions, we will get our desired result.

Remark: There are many key moments in the development of the theory of varieties (and later schemes) where category theory "saves the day" by giving precise meaning to geometric properties of varieties. This is one of the many extraordinary insights due to Grothendieck.

Definition: A pair of objects X and Y in a category $\mathcal{C}$ has a product, denoted $X \times Y$, if $X \times Y$ is another object of $\mathcal{C}$, equipped with "projections:"

$$\pi_X : X \times Y \to X \text{ and } \pi_Y : X \times Y \to Y$$

that are universal in the sense that given any object Z and morphisms:

$$p : Z \to X \text{ and } q : Z \to Y$$

there is a **unique** morphism, which we will denote $(p, q) : Z \to X \times Y$ with the property that $\pi_X \circ (p, q) = p$ and $\pi_Y \circ (p, q) = q$.

Observation: Any two product objects are naturally isomorphic. If $(X \times Y)'$ is another product with projections π'_X and π'_Y, then the unique morphisms $(\pi'_X, \pi'_Y) : (X \times Y)' \to X \times Y$ and $(\pi_X, \pi_Y) : X \times Y \to (X \times Y)'$ are inverses to one another by the uniqueness in the definition.

Examples: (a) The product of a pair of sets is the Cartesian product.

(b) The product of a pair of topological spaces is the Cartesian product with the product topology.

(c) The product of a pair of differentiable manifolds is a patched product (i.e. if $X = \cup U_i$ and $Y = \cup V_j$ then $X \times Y = \cup(U_i \times V_j)$ with product homeomorphisms to open sets in $\mathbf{R}^{n+m}$). $X \times Y$ has the product topology.

Proposition 6.1: Every pair of affine varieties has a product affine variety.

Proof: Given $X \subseteq \mathbf{C}^m$ with $\mathbf{C}[X] = \mathbf{C}[x_1, ..., x_m]/P$ and $Y \subseteq \mathbf{C}^n$ with $\mathbf{C}[Y] = \mathbf{C}[y_1, ..., y_n]/Q$, consider the Cartesian product:

$$X \times Y \subseteq \mathbf{C}^{m+n}$$

First of all, $X \times Y$ is a closed set. If $P = \langle \{f_i\}_{i \in I} \rangle$ for $f_i \in \mathbf{C}[x_1, ..., x_m]$ and $Q = \langle \{g_j\}_{j \in J} \rangle$ for $g_j \in \mathbf{C}[y_1, ..., y_n]$, then $X \times Y = V(\langle \{f_i\}_{i \in I} \cup \{g_j\}_{j \in J} \rangle)$. Secondly, $X \times Y$ is irreducible. If $X \times Y = Z_1 \cup Z_2$ is a union of closed sets, then for each $a \in X$, the variety $\{a\} \times Y \cong Y$ is a union of closed sets:

$$\{a\} \times Y = (Z_1 \cap (\{a\} \times Y)) \cup (Z_2 \cap (\{a\} \times Y))$$

so since Y is irreducible, either $\{a\} \times Y \subseteq Z_1$ or $\{a\} \times Y \subseteq Z_2$. But the sets

$$X_i := \{a \in X \mid \{a\} \times Y \subseteq Z_i\} \subseteq X$$

are also closed since $X_i = \cap_{b \in Y}(Z_i \cap (X \times \{b\}))$ thought of as an intersection of subsets of X. Since X is irreducible and $X_1 \cup X_2 = X$, it follows that either $X_1 = X$ (and then $Z_1 = X \times Y$) or else $X_2 = X$ (and then $Z_2 = X \times Y$).

The two projections $\pi_X(a, b) = a$ and $\pi_Y(a, b) = b$ are regular maps by Proposition 3.5, and they have the desired universal property, since any regular maps $p : Z \to X$ and $q : Z \to Y$ are given by polynomials $h_1, ..., h_m$ and $k_1, ..., k_n$, respectively (Proposition 3.5 again), and then the (unique) map to the Cartesian product $(p, q) = (h_1, ..., h_m, k_1, ...k_n)$ is also regular.

Remark: Invoking the Nullstellensatz, we have proved that

$$\sqrt{\langle \{f_i\} \cup \{g_j\} \rangle} = I(X \times Y) \subset \mathbf{C}[x_1, .., x_m, y_1, ..y_n]$$

is prime. The ideal $\langle \{f_i\} \cup \{g_j\} \rangle$ itself is prime, but this is left as an exercise.

Proposition 6.2: Every pair of varieties has a product variety.

Proof: We nearly proved this already for a pair of projective spaces $\mathbf{CP}^m, \mathbf{CP}^n$ in Proposition 5.4. The Segre "variety" $S_{m,n}$ has the desired properties, but we haven't actually proved that $S_{m,n}$ is irreducible. To see this, as in Proposition 6.1, we only need to know that the sets:

$$\{a\} \times \mathbf{CP}^n \subset S_{m,n} \text{ and } \mathbf{CP}^m \times \{b\} \subset S_{m,n}$$

are closed and isomorphic to $\mathbf{CP}^n$ and $\mathbf{CP}^m$ respectively. But not only are they closed, they are intersections of $S_{n,m}$ with the **linear** subspaces of $\mathbf{CP}^{(m+1)(n+1)-1}$ with respective ideals:

$$\langle\{a_i x_{kl} - a_k x_{il}\}\rangle \quad \text{and} \quad \langle\{b_j x_{kl} - b_l x_{kj}\}\rangle$$

and it follows that the bijections $\mathbf{CP}^n \leftrightarrow \{a\} \times \mathbf{CP}^n$ and $\mathbf{CP}^m \leftrightarrow \mathbf{CP}^m \times \{b\}$ are isomorphisms. For a general pair of projective varieties $X \subseteq \mathbf{CP}^m$ and $Z \subseteq \mathbf{CP}^n$, let $I(X) = \langle F_1, ..., F_r\rangle \in \mathbf{C}[x_0, ..., x_m]$ and $I(Z) = \langle G_1, ..., G_s\rangle \in \mathbf{C}[y_0, ..., y_n]$. Then the Cartesian product $X \times Z \subseteq S_{m,n}$ is the closed set:

$$V(\langle\{\overline{F}_\alpha(x_{0l}, ..., x_{ml}), \overline{G}_\beta(x_{i0}, ..., x_{in})\}\rangle) \subset S_{m,n}$$

And if $W \subseteq X$ and $Y \subseteq Z$ are open subsets (i.e. quasi-projective varieties) whose complements have ideals $\langle H_\gamma\rangle$ and $\langle K_\delta\rangle$, respectively, then:

$$W \times Y = X \times Z - (X \times V(\langle K_\delta\rangle) \cup (V(\langle H_\gamma\rangle) \times Z) =$$

$$X \times Z - V(\langle\{H_\gamma(x_{0l}, ..., x_{nl})K_\delta(x_{i0}, ..., x_{im})\}\rangle)$$

and irreduciblity is proved as for projective space, and the universal property with respect to regular maps follows immediately from Proposition 4.4.

Before we get to some peculiarities and examples of this product, here is a consistency result which isn't immediately obvious.

Corollary 6.3: The products of affine varieties in Propositions 6.1 and 6.2 yield isomorphic quasi-projective varieties.

Proof: By the universal property, all we need to do is to prove that the quasi-projective product from Proposition 6.2 is an affine variety! But if $W = \overline{W} - V(x_0)$ and $Y = \overline{Y} - V(y_0)$ as in Proposition 4.7, then:

$$W \times Y = \overline{W} \times \overline{Y} - V(\langle\{\overline{x_{0l}x_{i0}}\}\rangle)$$

by Proposition 6.2. But the equations $x_{00}x_{il} = x_{0l}x_{i0}$ (for the Segre variety) tell us $V(\langle\{\overline{x_{0l}x_{i0}}\}\rangle) = V(\overline{x}_{00})$ so $W \times Y = \overline{W} \times \overline{Y} - V(x_{00})$, is isomorphic to an affine variety by Proposition 4.7.

Examples: (a) In the quadric $\mathbf{CP}^1 \times \mathbf{CP}^1 = V(x_{00}x_{11} - x_{01}x_{10}) \subset \mathbf{CP}^3$, each further $V(\overline{x}_{ij})$ is a pair of intersecting projective lines on the quadric:

$$V(\overline{x}_{00}) = \{(0 : a_{01} : 0 : a_{11})\} \cup \{(0 : 0 : a_{01} : a_{11})\}$$

$$V(\overline{x}_{01}) = \{(a_{00} : 0 : a_{01} : 0)\} \cup \{(0 : 0 : a_{01} : a_{11})\}$$
$$V(\overline{x}_{10}) = \{(a_{00} : a_{01} : 0 : 0)\} \cup \{(0 : a_{01} : 0 : a_{11})\}$$
$$V(\overline{x}_{11}) = \{(a_{00} : a_{01} : 0 : 0)\} \cup \{(a_{00} : 0 : a_{01} : 0)\}$$

and

$$\mathbf{CP}^1 \times \mathbf{CP}^1 - V(\overline{x}_{00}) = \{(1 : a_{01} : a_{10} : a_{01}a_{10})\}$$

(setting $a_{00} = 1$ and using the equation) is isomorphic to $\mathbf{C}^2 = \mathbf{C}^1 \times \mathbf{C}^1$.

(b) On the other hand, the intersection of the quadric with a plane $V(\sum c_{ij}x_{ij})$ is **not**, in general, a pair of intersecting lines. For instance:

$$C_1 := V(\overline{x_{00} - x_{11}}) - V(\overline{x}_{00}) = \{(1 : a_{01} : \frac{1}{a_{01}} : 1)\}$$

is a **hyperbola**,and $V(\overline{x_{00} - x_{11}}) = C_1 \cup \{(0 :1 :0 :0)\} \cup \{(0: 0 :1 :0)\}$ and

$$C_2 := V(\overline{x_{10} - x_{01}}) - V(\overline{x}_{00}) = \{(1 : a_{01} : a_{01} : a_{01}^2)\}$$

is a **parabola** (in $\mathbf{C}^2 = \{(1 :a :a :b)\}$) with $V(\overline{x}_{10} - \overline{x}_{01}) = C_2 \cup \{(0 :0 :0 :1)\}$

(c) According to the prescription of Proposition 6.2, the equations for the point $(1 : 0) \times (1 : 0) = V(x_1) \times V(y_1) \in \mathbf{CP}^1 \times \mathbf{CP}^1 \subset \mathbf{CP}^3$ are:

$$V(x_{00}x_{11} - x_{10}x_{01}, x_{10}, x_{11}, x_{01}, x_{11}) = (1 : 0 : 0 : 0)$$

which are more equations than we need, but that doesn't matter.

Crucial Fact: Products of varieties do not have the product topology.

Recall that a set $Z \subset X \times Y$ is closed for the product topology when Z is an intersection of sets of the form $Z_X \times Z_Y$. In $\mathbf{C}^1$, for example, the closed sets are all finite (or $\mathbf{C}^1$), so the only closed sets in $\mathbf{C}^2$ for the product topology would be of the form $(S_X \times Y) \cup (X \times S_Y) \cup S$ where S_X, S_Y and S are all finite. But there are plenty of other closed sets in the Zariski topology, such as the diagonal:

$$\Delta = V(x - y) \subset \mathbf{C}^2$$

Definitions: Let Y be an object in a category of topological spaces where products always exist and are always Cartesian products (as sets).

(a) Consider the *diagonal map* $\delta : Y \to Y \times Y$ applying the universal property to the pair of identity morphisms $p, q : Y \to Y$, and let $\Delta := \delta(Y)$. Then $\Delta \subset Y \times Y$ is the *diagonal*, and Y is *separated* if $\Delta \subset Y \times Y$ is closed.

(b) Y is *proper* if Y is separated and if for every object W, the projection morphism $\pi_W : W \times Y \to W$ is closed (i.e. maps closed sets to closed sets).

Proposition 6.4: Y as above is separated if and only if for every X and pair of morphisms $p, q : X \to Y$, the following subset is closed:

$$\{p = q\} := \{x \in X \mid p(x) = q(x)\} \subseteq X$$

Proof: Notice first that $\Delta = \{\pi_1 = \pi_2\} \subset Y \times Y$, for the two projections $\pi_i : Y \times Y \to Y$. Next, notice that the unique morphism $(p, q) : X \to Y \times Y$ has the property that $\{p = q\} = (p, q)^{-1}(\Delta)$. If Δ is closed, then this is always closed since the morphisms are all continuous.

Proposition 6.5: (a) If products have the product topology in the category, then Y is Hausdorff if and only if Y is separated.

(b) Quasi-projective varieties are always separated.

Proof: (a) Hausdorff means that if $x \neq y \in Y$ then there are open sets $x \in U_x, y \in U_y \subset Y$ with empty intersection. If Y is Hausdorff and $Y \times Y$ has the product topology, then $U_x \times U_y$ is open, so $Y \times Y - \Delta$ is covered by open sets $U_x \times U_y$ and Δ is closed. Conversely, if $Y \times Y$ has the product topology and $x \neq y \in Y$, then if Δ is closed, it follows that $(x, y) \in U \subset Y \times Y - \Delta$ and U is a union of open sets of the form $U_1 \times U_2$. One (or more) of these contains (x, y), which then can be declared to be $U_x \times U_y$.

(b) It suffices to prove $\mathbf{CP}^n$ is separated, since once $\Delta \subset \mathbf{CP}^n \times \mathbf{CP}^n$ is closed, then the diagonal in $Y \times Y$ is closed for any $Y \subset \mathbf{CP}^n$ since $Y \times Y$ has the induced topology. But the diagonal is given by explicit equations:

$$\Delta = V(\langle\{\overline{x_{ij} - x_{ji}}\}\rangle) \subset S_{n,n} = V(\{x_{ij}x_{kl} - x_{il}x_{kj}\}) \subset \mathbf{CP}^{2n+1}$$

so of course it is closed.

Example: The closure of the parabola in the earlier example is the diagonal:

$$\Delta = V(\overline{x_{01} - x_{10}}) \subset V(x_{00}x_{11} - x_{01}x_{10}) = \mathbf{CP}^1 \times \mathbf{CP}^1 \subset \mathbf{CP}^3$$

and $V(\overline{x_{01} - x_{10}}) = V(x_{00}x_{11} - z^2)$ is the smooth projective conic from §4 (in the plane $\mathbf{CP}^2 \cong V(x_{01} - x_{10})$ with coordinates $x_{00}, x_{11}, z = x_{01} = x_{10}$).

Non-example: If X is any topological space together with a sheaf $\mathcal{O}_X$ of regular functions that admits an open cover by finitely many affine varieties, then X is often called a *prevariety*. A quasi-projective variety is one example, but there are other prevarieties. For example, the "affine line with 2 origins:"

$$X := (\mathbf{C}^1 - \{0\}) \cup \{0'\} \cup \{0''\}$$

can be given a topology and sheaf $\mathcal{O}_X$ so that the subsets $U' = X - \{0''\}$ and $U'' = X - \{0'\}$ are open and isomorphic to $\mathbf{C}^1$ with $0'$ (respectively $0''$) replacing 0 as the origin. X is not separated because (Proposition 6.4) the maps $p : \mathbf{C}^1 \xrightarrow{\sim} U' \subset X$ and $q : \mathbf{C}^1 \xrightarrow{\sim} U'' \subset X$ give $\{p = q\} = \mathbf{C}^1 - \{0\}$, which isn't closed in $\mathbf{C}^1$. So X is not isomorphic to a quasi-projective variety.

Let's begin our discussion of properness with a simple observation:

Proposition 6.6: If $Y \subset \mathbf{CP}^n$ is a quasi-projective variety and $Y \neq \overline{Y}$, then Y isn't proper.

Proof: Recall that all quasi-projective varieties are separated. If $Y \neq \overline{Y}$, consider:

$$Z := \Delta \cap (\overline{Y} \times Y) \subset \overline{Y} \times Y$$

Then Z is closed in $\overline{Y} \times Y$ but $\pi_{\overline{Y}}(Z) = Y \subset \overline{Y}$ isn't closed in $\overline{Y}$ because it is an open subset and $\overline{Y}$ is irreducible. So Y isn't proper.

Remark: Recall that a basic open subset $U \subset Y$ of an affine variety is affine (Proposition 3.6). When we prove below that projective varieties are proper, it follows from Proposition 6.6 that **no** open subset $U \subset X$ of a projective variety (except X itself) is isomorphic to a projective variety. For this reason, proper varieties are often called *complete*.

The two notions of compactness and properness agree when products have the product topology, once we dispose of some pathological cases. For example, in the category of sets with the **discrete** topology, all objects are proper because all subsets are closed, but no such infinite set is compact!

Proposition 6.7: (a) If products in a given category have the product topology, then compact implies proper. If there some object W with a chain $W \supseteq W_1 \supseteq \cdots$ of open sets so that $\cap W_i \subset W$ isn't open, and if every open cover of Y has a countable subcover, then proper implies compact for Y.

(b) Projective varieties are always proper.

Proof of (a) Y is compact if it is Hausdorff and if every open cover $Y = \cup_{\lambda \in \Lambda} U_\lambda$ has a finite subcover. Hausdorff and separated agree by Proposition 6.5. Given a compact Y and a closed subset $Z \subset Y \times W$, pick $w \in W - \pi_W(Z)$. Then for each point $y \in Y$, we can find a product neighborhood $U_y \times W_y$ of (y, w) which is disjoint from the closed set Z. Thus $Y = \cup_{y \in Y} U_y$, so since Y is compact, a finite number $U_{y_1}, ..., U_{y_r}$ will also cover, and it follows that $w \in \cap_{i=1}^r W_{y_i}$, and moreover that this open subset of W is in the complement of $\pi_W(Z)$. Thus $\pi_W(Z)$ is closed and Y is proper.

Conversely, suppose Y is proper and that the extra conditions hold. Given an open cover $Y = \cup_{\lambda \in \Lambda} U_\lambda$, take a countable subcover $Y = \cup_{n=1}^\infty U_{\lambda_n}$ and consider the ascending chain of open sets $V_n = U_{\lambda_1} \cup ... \cup U_{\lambda_n}$. We want to prove that this chain is eventually stationary, filling Y. In the product $Y \times W$ (for the W in the Proposition) consider the set $Z := Y \times W - \cup_{n=1}^\infty V_n \times W_n$. I claim that if V_n is not eventually Y, then $\pi_W(Z) = W - \cap W_i$, which isn't closed, contradicting the properness of Y. To see this, suppose that $w \in \cap W_n$. For each $y \in Y$, we can find a V_n from ascending chain so that $y \in V_n$, and then $(w, y) \in W_n \times V_n$ so $(w, y) \notin Z$. Since we can do this for every y, it follows that $w \notin \pi_W(Z)$ and $\pi_W(Z) \subseteq W - \cap W_n$. On the other hand, if $w \notin W_n$ for some (minimal) n and if the increasing chain of V's isn't eventually stationary, it follows in particular that that there is a $y \in Y - V_n$ for this n, and that $y \notin V_m$ for all $m \leq n$, so for this choice of y, $(w, y) \notin W_n \times V_n$ for all n. Thus $(w, y) \in Z$ and $w \in \pi_W(Z)$.

Remark: The conditions in (a) are very mild. In most reasonable settings, there are chains of open sets $W \supseteq W_1 \supseteq ...$ with $\cap W_n = \{w\}$ for any $w \in W$, so unless points are open (the discrete topology case!) almost any W will do. And every open cover has a countable subcover in most reasonable settings, such as open subsets of $\mathbf{R}^n$ or manifolds patched from finitely many of them.

For the proof of (b), we will need a new result from commutative algebra which will have many "geometric" consequences when applied to varieties. There will be several versions of this result. Here is the first one:

Nakayama's Lemma I: Suppose:

- A is a commutative ring with 1,
- M is a finitely generated A-module, and
- $I \subseteq A$ is an ideal that stabilizes M, in the sense that $M = IM$.

Then there is an $a = 1 + b \in A$ with $b \in I$ such that $am = 0$ for all $m \in M$. (i.e. a annihilates the module M).

Proof: Let $m_1, ..., m_n$ generate M. By assumption, we can solve:

$$m_i = \sum_{j=1}^{n} b_{ij} m_j$$

with $b_{ij} \in B$. This tells us that the matrix $I_n - B$ annilates each m_j, hence $(I_n - B)m = 0$ for each $m \in M$ (writing $m = \sum a_j m_j$). By Cramer's rule,

$$a = \det(I_n - B)$$

satisfies $am = 0$ for all $m \in M$, and evidently $a = 1 + b$ for some $b \in I$.

Proof of (b) (Due to Grothendieck) If $X \subset \mathbf{CP}^n$ is a projective variety, then closed subsets of $W \times X$ are closed in $W \times \mathbf{CP}^n$, so it suffices to prove that $\mathbf{CP}^n$ is proper. We may also assume W is affine. If $\pi_W(Z)$ weren't closed for some $Z \subset W \times \mathbf{CP}^n$, then $\pi_U(Z \cap (U \times \mathbf{CP}^n)) = \pi_W(Z) \cap U$, wouldn't be closed in U for some U in an affine open cover of W. Finally, if $W \subset \mathbf{C}^m$ is closed, we may replace W by $\mathbf{C}^m$, since subsets of $W \times \mathbf{CP}^n$ and W are closed if and only if they are closed in $\mathbf{C}^m \times \mathbf{CP}^n$ and $\mathbf{C}^m$ respectively.

We will label points of $\mathbf{C}^m \times \mathbf{CP}^n$ by:

$$(\underline{a}, \underline{b}) := ((a_1, ..., a_m), (b_0 : ... : b_n)) \in \mathbf{C}^m \times \mathbf{CP}^n$$

and the coordinate rings by $\mathbf{C}[\underline{x}] = \mathbf{C}[x_1, ..., x_m]$ and $\mathbf{C}[\underline{y}] = \mathbf{C}[y_0, ..., y_n]$. We are trying to show that if $Z \subset \mathbf{C}^m \times \mathbf{CP}^n$ is closed, then:

$$\pi_{\mathbf{C}^m}(Z) = \{\underline{a} \in \mathbf{C}^m \mid (\underline{a}, \underline{b}) \in Z \text{ for some } \underline{b} \in \mathbf{CP}^n\} \subset \mathbf{C}^m$$

is also closed. We will do this in two steps.

Step 1: Define a partial grading on $\mathbf{C}[\underline{x}, \underline{y}]$ by degree in the y-variables, so:

$$\mathbf{C}[\underline{x}, \underline{y}] = \bigoplus_{d=0}^{\infty} \mathbf{C}[\underline{x}, \underline{y}]_d := \bigoplus_{d=0}^{\infty} \mathbf{C}[\underline{x}] \otimes_{\mathbf{C}} \mathbf{C}[\underline{y}]_d$$

and homogeneous polynomials and ideals in the usual way for this grading. For each $F \in \mathbf{C}[\underline{x}, \underline{y}]_d$, it makes sense to ask whether $F(\underline{a}, \underline{b}) = 0$ or not, so we may define $V(I) \subset \mathbf{C}^m \times \mathbf{CP}^n$ for homogeneous ideals I. Our first step is to show that every closed subset $Z \subset \mathbf{C}^m \times \mathbf{CP}^n$ is a $V(I)$ for some such I.

Proof: Cover $\mathbf{CP}^n$ by open sets $U_i = \mathbf{C}^m \times \mathbf{C}^n$, with coordinate rings:

$$\mathbf{C}[U_i] \cong \mathbf{C}[x_1, ..., x_m, \frac{y_0}{y_i}, ..., \frac{y_n}{y_i}],$$

Define the homogeneous ideal I associated to Z by:

$$I_d := \{F \in \mathbf{C}[\underline{x}, \underline{y}]_d \mid \frac{F}{y_i^d} \in I(Z \cap U_i) \subset \mathbf{C}[U_i] \text{ for } \textbf{all } i\}$$

and define $I := \sum_{d=0}^{\infty} I_d$. This is the homogeneous ideal we will use. It is immediate from the definition that $Z \subseteq V(I)$, since $\frac{F}{y_i^d} \in I(Z \cap U_i)$ if and only if F vanishes at every point of $Z \cap U_i$.

For the other inclusion, take $f \in I(Z \cap U_i)$. Pick d large enough so that $G := y_i^d f \in \mathbf{C}[\underline{x}, \underline{y}]_d$. We do not know that $G \in I_d$, but each of the polynomials $f_j := \frac{G}{y_j^d}$ vanishes on the set $Z \cap U_i \cap U_j$, so $\frac{y_i}{y_j} f_j \in I(Z \cap U_j)$ because this function vanishes on $Z \cap U_i \cap U_j$ and on $Z \cap (U_j - U_i)$. So for any $e > d$, we do have $F = y_i^e f \in I_e$ satisfying $F(x_1, ..., x_m, \frac{y_0}{y_i}, ..., \frac{y_n}{y_i}) = f$. If $(\underline{a}, \underline{b}) \notin Z$ then $(\underline{a}, \underline{b}) \in U_i$ for some i, and so there is an $f \in I(Z \cap U_i)$ such that $f(\underline{a}, \frac{b_0}{b_i}, ..., \frac{b_n}{b_i}) \neq 0$. Any $F \in I_e$ constructed above will also satisfy $F(\underline{a}, \underline{b}) \neq 0$, so we have proved that $Z = V(I)$.

Step 2. For the homogeneous ideal I from Step 1,

$$\pi_{\mathbf{C}^m}(Z) = V(I_0)$$

so in particular the projection is closed, proving (b).

Proof: Again, it is clear that $\pi_{\mathbf{C}^m}(Z) \subseteq V(I_0)$. Suppose $\underline{a} \notin \pi_{\mathbf{C}^m}(Z)$ and let $m_{\underline{a}} = \langle x_1 - a_1, ..., x_m - a_m \rangle$. The closed set $\pi_{\mathbf{C}^m}^{-1}(\underline{a}) = \{\underline{a}\} \times \mathbf{CP}^n$ is disjoint from Z, so intersecting with U_i gives disjoint closed sets $Z \cap U_i$ and $\{\underline{a}\} \times U_i$ in U_i with ideals $I(Z \cap U_i)$ and $\langle m_{\underline{a}} \rangle \subset \mathbf{C}[U_i]$ respectively. Thus

$$I(Z \cap U_i) + \langle m_{\underline{a}} \rangle = \mathbf{C}[U_i]$$

by the Hilbert Nullstellensatz. In other words, for each $i = 0, ..., n$ there exist $f_i \in I(Z \cap U_i), m_{i,j} \in m_{\underline{a}}$ and $g_{i,j} \in \mathbf{C}[U_i]$ such that $f_i + \sum_j g_{i,j} m_{i,j} = 1$.

Multiplying through by sufficiently large powers e_i of the y_i, we get:

$$F_i + \sum_j G_{i,j} m_{i,j} = y_i^{e_i} \text{ for } F_i \in I_{e_i}, G_{i,j} \in \mathbf{C}[\underline{x}, \underline{y}]_{e_i}$$

(and $F_i \in I_{e_i}$ as in Step 1). If we further take $d \geq \sum e_i$, then this gives us

$$I_d + m_{\underline{a}} \mathbf{C}[\underline{x}, \underline{y}]_d = \mathbf{C}[\underline{x}, \underline{y}]_d$$

since every monomial of degree d contains at least one y_i to the power e_i. Now consider the $\mathbf{C}[\underline{x}]$-module $M := \mathbf{C}[\underline{x}, \underline{y}]_d / I_d$. The equality above tells us $m_{\underline{a}} M = M$, hence by Nakayama's Lemma there is an $f \in \mathbf{C}[\underline{x}] - m_{\underline{a}}$ that annihilates M, i.e. so that $f\mathbf{C}[\underline{x}, \underline{y}]_d \subseteq I_d$. Thus $f y_i^d \in I_d$ for all i and it follows that $f \in I_0$. Thus we have found an $f \in I_0$ with $f(\underline{a}) \neq 0$, as desired.

Example: In $\mathbf{C}^2 \times \mathbf{CP}^1$, consider:

$$Z = V(\langle x_1 y_1 - y_0 x_2, \quad y_0 y_1 \rangle)$$

When we write this as a union of the two open sets U_0, U_1, we get:

$$Z = \{(0, a_2, 1, 0)\} \cup \{(a_1, 0, 0, 1)\}$$

so $\pi_{\mathbf{C}^2}(Z) = V(x_1 x_2)$ which isn't readily apparent from the equations for Z. But the point is that:

$$x_1^2(y_0 y_1) - y_0 x_1 (x_1 y_1 - y_0 x_2) = y_0^2 x_1 x_2, \text{ and}$$

$$x_2^2(y_0 y_1) + y_1 x_2 (x_1 y_1 - y_0 x_2) = y_1^2 x_1 x_2$$

so $x_1 x_2 \in I_0$.

Exercises 6.

1. If $\mathbf{C}[X] = \mathbf{C}[x_1, ..., x_m]/\langle\{f_i\}\rangle$ and $\mathbf{C}[Y] = \mathbf{C}[y_1, ..., y_n]/\langle\{g_j\}\rangle$ as in Proposition 6.1, finish the proof that $I(X \times Y) = \langle\{f_i\} \cup \{g_j\}\rangle$ and conclude that:

$$\mathbf{C}[X \times Y] \cong \mathbf{C}[X] \otimes_{\mathbf{C}} \mathbf{C}[Y]$$

for any pair X, Y of affine varieties.

2. If $X \subset \mathbf{CP}^n$ is a projective variety, then the *affine cone* $C(X) \subset \mathbf{C}^{n+1}$ is the union of the lines parametrized by the points of X. In other words:

$$(b_0, ..., b_n) \in C(X) \Leftrightarrow (b_0 : ... : b_n) \in X \text{ or } (b_0, ..., b_n) = (0, ..., 0)$$

Prove that if x_i is chosen so that $X \not\subset V(x_i)$, then:

$$C(X) - V(x_i) \cong (X - V(x_i)) \times (\mathbf{C}^1 - 0)$$

hence that $C(X) - 0$ is covered by such open affine sets.

3. (a) Prove that $\mathbf{CP}^m \times \mathbf{CP}^n \neq \mathbf{CP}^{m+n}$ when $m, n > 0$.

(b) Given homogeneous coordinate rings $\mathbf{C}[x_0, ..., x_n]$ and $\mathbf{C}[y_0, ..., y_n]$ on $\mathbf{CP}^m$ and $\mathbf{CP}^n$, prove that the closed subsets of $\mathbf{CP}^m \times \mathbf{CP}^n$ are all of the form $V(I)$, where $I \subset \mathbf{C}[x_0, ..., x_m, y_0, ..., y_m]$ is a *bihomogeneous ideal*, i.e. I is generated by bihomogeneous polynomials $F(\underline{x}, \underline{y})$ (of bidegrees (d, e)) for the double grading:

$$\mathbf{C}[x_0, ..., x_m, y_0, ..., y_m] = \bigoplus_{d=0}^{\infty} \bigoplus_{e=0}^{\infty} \mathbf{C}[x_0, ..., x_m]_d \otimes_{\mathbf{C}} \mathbf{C}[y_0, ..., y_n]_e$$

(c) Express the twisted cubic curve as a bihomogeneous hypersurface in $\mathbf{CP}^1 \times \mathbf{CP}^1$.

(d) Which of the irreducible "bihomogeneous hypersurfaces" $V(F) \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ are intersections of $\mathbf{CP}^1 \times \mathbf{CP}^1 = S_{1,1} \subset \mathbf{CP}^3$ with hypersurfaces in $\mathbf{CP}^3$?