Complex Algebraic Geometry: An Overview
Aaron Bertram, 2010

Complex Algebraic Geometry is concerned with:

(i) Finite field extensions:

$$\mathbb{C}(x_1, \ldots, x_n) \subset K$$

of the fields of rational functions in n variables over $\mathbb{C}$ and:

(ii) the smooth complex projective varieties X that "model" K, in the sense that:

$$\mathbb{C}(X) \cong K$$

is the "field of rational functions" on X.

Remarks: (i) The number of variables is the *transcendence degree* of K (as a field extension of $\mathbb{C}$) and is an invariant of K.

(ii) It is a non-trivial theorem, originally due to Hironaka, that a smooth projective model exists for every such field K.

This begs the following questions:

Question 1: Why the complex numbers and not some other field, or even commutative ring (e.g. $\mathbb{Z}$) of coefficients?

Question 2: What is a variety, and what is the advantage in having it as a model of a field?

The answer to Question 1 is that there is, in fact, no necessity to limit ourselves to doing algebraic geometry over the field of complex numbers (or indeed over any particular field). Grothendieck's theory of schemes is well suited to the study of both individual varieties and of *families* of varieties, as well as other more exotic objects that allow one, for example, to explore infinitessimal properties of varieties. However, the machinery involved in Grothendieck's theory is substantial, and we will refer to it only as needed. There are several very good reasons, however, for focusing on the field of complex numbers:

• The complex numbers are an algebraically closed field; polynomials in one variable always have roots, and similarly, polynomials in many variables have loci of zeroes with predictable properties.

• The complex numbers are a field of characteristic zero. This means, for example, that the derivative of a polynomial is identically zero if and only if the polynomial is a constant. In the same vein, it also means that there are no inseparable field extensions.

• The locus of zeroes of a system of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ has an "analytic" structure, allowing for simultaneous study by algebraic, analytic and topological techniques.

As for Question 2, a **variety** is a "geometric" object, consisting of:

(a) A set X with a (Zariski) topology with some unusual properties,

(b) A sheaf $\mathcal{O}_X$ of (finitely generated) $\mathbb{C}$-algebras on X

that is *locally affine* and *separated* (which substitutes for Hausdorff).

Two basic examples of varieties are:

Affine varieties, whose points are in bijection with the maximal ideals of an integral domain $\mathbb{C}[X]$ that is finitely generated as $\mathbb{C}$-algebra, and

Projective varieties, whose points are in bijection with the "nearly maximal" homogeneous ideals of a *graded* integral domain $R^\bullet(X)$ with $R^0(X) = \mathbb{C}$ that is finitely generated as a $\mathbb{C}$-algebra.

Back to Question 2, every point $x \in X$ of a variety has a canonical *Zariski tangent space* $T_x X$ that is finite-dimensional as a vector space over $\mathbb{C}$, of dimension at least as great as the transcendence degree of $\mathbb{C}(X)$, which is the dimension of X. Moreover, the set of points of X for which the Zariski tangent space is equal to the dimension of X is open and dense in (the Zariski topology on) X. Such points are **non-singular** points of X, and X is *smooth* if all its points are non-singular.

Smooth projective models X for K have the following nice properties:

• They vary in families, leading to a well-defined notion of "moduli."

• They are compact complex analytic manifolds, so they are oriented and have a graded cohomology ring that satisfies Poincaré duality and is equipped with a "Hodge structure" that extends to families.

• They are uniquely determined in dimension one, and, though they are not unique in dimension two, there are essentially unique "minimal" models for most fields. However, in dimensions three and more, one needs to allow singularities in order to find a minimal model.

• They can be used to define algebraic invariants, e.g. plurigenera and irregularity, via spaces of (pluri)-differential forms that are defined in terms of a particular smooth projective model, and then proven to be independent of the choice of model. These are (among) the discrete invariants of a field K that are not at all evident without invoking the projective models.

This semester, I want to focus on specific examples, namely curves, surfaces and toric varieties, though we will need to build upon some of the general theory (e.g. from Hartshorne's book), including:

- Varieties and schemes (separatedness and properness)
- Divisors and linear series
- Line bundles and vector bundles
- Coherent sheaves and sheaves of differentials
- Blowing up
- Cohomology of coherent sheaves
- Serre duality
- Flatness and families

Curves and their Jacobians: Not necessarily in this order,

- The genus of a curve
- Degrees of divisors
- Riemann-Hurwitz and Riemann-Roch for curves
- Degeneracy loci and special divisors
- The Jacobian of a curve
- Why is the Jacobian an abelian variety?
- Abel's theorem
- Riemann's singularity theorem
- The Hurwitz "scheme"
- Petri's explicit equations for a canonical curve
- Tropical curves?!

References:

Mumford, *Curves and their Jacobians*,
Arbarello -Cornalba-Griffiths-Harris, *Geometry of algebraic curves*

Surfaces:

- The intersection form on divisors
- Riemann-Roch for surfaces
- Ruled and rational surfaces
- Castelnuovo's criterion
- Lüroth's theorem
- Kodaira dimension
- Introducing K3 surfaces
- First steps towards classification

Reference: Beauville, *Complex algebraic surfaces*

Toric varieties:

- Lots of cool stuff

Reference: Fulton, *Toric varieties*