Math 6130 Notes. Fall 2002.

9. Normal Varieties. Let's get back to our original motivation...to find an analogue in algebraic geometry of the ring of integers of a number field. That is, we want to fill in a finite field extension $\mathbf{C}(x_1, ..., x_n) \subset K$:

$$\begin{array}{ccc} \mathbf{C}[X] & \subset & K \\ \cup & & \cup \\ \mathbf{C}[x_1, ..., x_n] & \subset & \mathbf{C}(x_1, ..., x_n) \end{array}$$

uniquely with the coordinate ring of a affine variety X such that $K = \mathbf{C}(X)$.

In number theory, one fills in with a Dedekind domain. We will fill in with the *integral closure* of $\mathbf{C}[x_1, ..., x_n]$ in K. We will see that integral closures are coordinate rings of "normal" affine varieties, and that any affine (or quasi-projective) variety Y can be canonically normalized in a finite field extension $\mathbf{C}(Y) \subset K$, i.e. there is a normal variety X with $\mathbf{C}(X) = K$ and a finite map $\Phi : X \to Y$. In this section, we will explore normal varieties and show how Zariski's main theorem follows from Grothendieck's Theorem (§7).

Definition: For an inclusion $A \subset K$ of a Noetherian domain A in a field K,

(a) an element $\alpha \in K$ is *integral* over A if α is a root of some monic polynomial $x^d + a_{d-1}x^{d-1} + ... + a_0$ with coefficients in A, and

(b) the set of $\alpha \in K$ that are integral over A is the *integral closure* $\overline{A} \subset K$.

(c) A is *integrally closed* if $A = \overline{A}$ when K is the field of fractions of A.

Remark: Each $a \in A$ is the root of $x - a$, so $A \subseteq \overline{A}$ (for every K). If $A \subset K \subset L$, then $\overline{A}$ (in K) is contained in $\overline{A}$ (in L), so the integral closure of A in its field of fractions is contained in all other integral closures.

Proposition 9.1 Given $A \subset K$, then $\alpha \in K$ is integral over A if and only if $A[\alpha] \subset K$ is a finitely generated A-module.

Proof: If $\alpha \in K$ is integral over A, then α is a root of a polynomial $x^d + a_{d-1}x^{d-1} + ... + a_0$, and then $1, \alpha, ...\alpha^{d-1}$ generate $A[\alpha]$ as an A-module. Conversely, if $A[\alpha] \subset K$ is finitely generated as an A-module, then the chain $A \subseteq A + \alpha A \subseteq A + \alpha A + \alpha^2 A \subseteq .. \subseteq A[\alpha]$ must be eventually stationary, so $\alpha^d = -a_0 - a_1\alpha - ... - a_{d-1}\alpha^{d-1}$ for some d and $a_0, ..., a_{d-1} \in A$, and then α is a root of the monic polynomial $x^d + a_{d-1}x^{d-1} + ... + a_0$.

Corollary 9.2: Each integral closure $\overline{A} \subset K$ is a domain.

Proof: We need $\overline{A}$ to be closed under subtraction and multiplication. Once it is a ring, then it is a domain since it is contained in a field.

Given $\alpha, \beta \in K$ integral over A, then by Proposition 9.1, $A[\alpha]$ is a finitely generated A-module and a Noetherian domain and β is then integral over $A[\alpha] \subset K$, so by Proposition 9.1 again, $A[\alpha, \beta]$ is a finitely generated $A[\alpha]$-module, hence also finitely generated as an A-module.

Since $A[\alpha - \beta]$ and $A[\alpha\beta]$ are submodules of $A[\alpha, \beta]$, they must also be finitely generated A-modules by Proposition 1.2, and then $\alpha - \beta$ and $\alpha\beta$ are integral over A by Proposition 9.1, hence they both belong to $\overline{A}$.

Examples: (a) The integral closure of the ordinary integers $\mathbf{Z} \subset K$ in a finite extension of $\mathbf{Q}$ is called the ring of integers of K, often denoted $\mathcal{O}_K$.

(b) $\mathbf{C}[t^2, t^3]$ is not integrally closed. The rational function $t = \frac{t^3}{t^2} \in \mathbf{C}(t)$ is a root of the monic polynomial $x^2 - t^2$, so it is integral over $\mathbf{C}[t^2, t^3]$, but not contained in $\mathbf{C}[t^2, t^3]$. In fact, $\overline{\mathbf{C}[t^2, t^3]} = \mathbf{C}[t] \subset \mathbf{C}(t)$ (see Remark (a)).

(c) If $\Phi : X \to Y$ is a finite map of affine varieties, then $\mathbf{C}[X]$ is a finitely generated $\mathbf{C}[Y]$-module. By Proposition 9.1, each $\alpha \in \mathbf{C}[X]$ is integral over $\mathbf{C}[Y]$. So if $\mathbf{C}[X]$ is integrally closed, then $\mathbf{C}[X] = \overline{\mathbf{C}[Y]} \subset \mathbf{C}(X)$.

Remarks: (a) Every UFD A is integrally closed (in its field of fractions K).

Suppose $\frac{a}{b} \in K$ is in lowest terms and $\frac{a}{b}$ is a root of a monic polynomial $x^d + a_{d-1}x^{d-1} + ... + a_0$. Then $a^d = -b(a_{d-1} + ... + b^{d-1}a_0)$ so b divides a^d, which can only happen if b is a unit, i.e. $\frac{a}{b} \in A$.

(b) If $S \subset A$ is a multiplicative set, then $\overline{(A_S)} = (\overline{A})_S$ in any field K.

Suppose $\alpha \in K$ is a root of $x^d + a_{d-1}x^{d-1} + ... + a_0$ for $a_i \in A$. Then each $\frac{\alpha}{s} \in K$ is a root of $x^d + \frac{a_{d-1}}{s}x^{d-1} + ... + \frac{a_0}{s^d}$. So $\overline{(A_S)} \supseteq (\overline{A})_S$.

Conversely, if $\alpha \in K$ is a root of $x^d + \frac{a_{d-1}}{s_{d-1}}x^{d-1} + ... + \frac{a_0}{s_0}$, let $s = \prod s_i$. Then $s\alpha$ is a root of $x^d + \frac{sa_{d-1}}{s_{d-1}}x^{d-1} + ... + \frac{s^d a_0}{s_0}$, and the coefficients of this polynomial are all in A, so $\alpha = \frac{s\alpha}{s} \in (\overline{A})_S$, thus $\overline{(A_S)} \subseteq (\overline{A})_S$

Definition: An affine variety X is *normal* if $\mathbf{C}[X]$ is integrally closed.

Examples: (a) $\mathbf{C}^n$ is normal, since $\mathbf{C}[x_1, ..., x_n]$ is a UFD.

(b) $\mathbf{C}[X]$ is not normal if $X = V(y^2 - x^3) \subset \mathbf{C}^2$, since $\mathbf{C}[X] \cong \mathbf{C}[t^2, t^3]$.

Proposition 9.3: If X is any affine variety then any finite field extension $\mathbf{C}(X) \subset K$ of the field of rational functions on X fills in with:

$$\begin{array}{ccc} \mathbf{C}[Y] & \subset & K \\ \cup & & \cup \\ \mathbf{C}[X] & \subset & \mathbf{C}(X) \end{array}$$

where Y is a uniquely determined affine variety such that: (i) Y is normal, (ii) $\mathbf{C}(Y) = K$, and (iii) $\mathbf{C}[Y]$ is a finitely generated $\mathbf{C}[X]$-module

Proof: First, uniqueness. If $\mathbf{C}[Y]$ is a finitely generated $\mathbf{C}[X]$-module, then $\mathbf{C}[Y] \subseteq \overline{\mathbf{C}[X]}$. But if $\mathbf{C}(Y) = K$ and Y is normal, then $\mathbf{C}[Y] = \overline{\mathbf{C}[Y]}$ (in K) and so $\mathbf{C}[Y] = \overline{\mathbf{C}[X]}$ is the integral closure in K, hence unique.

To prove the existence of Y, let $A = \overline{\mathbf{C}[X]}$. Letting $S = \mathbf{C}[X] - 0$ in Remark (b) above, we see that A_S is the integral closure of $\mathbf{C}(X)$ in K, which is K itself, since $\mathbf{C}(X) \subset K$ is a finite extension of fields. So K is the field of fractions of A. Thus once we know that $A = \mathbf{C}[Y]$ for some affine variety Y, then Properties (i) and (ii) are immediate. We will prove below that A is a finitely generated $\mathbf{C}[X]$-module, which will give us Property (iii) **and** the fact that $A = \mathbf{C}[Y]$, since a domain A that is a finitely generated module over a ring of the form $\mathbf{C}[x_1, ..., x_n]/P$ is itself of the form $\mathbf{C}[x_1, ..., x_n, y_1, ..., y_m]/Q$.

To prove finite generatedness, we take another field theory interlude.

Field Theory III: If $K \subset L$ is a finite field extension, then:

$$\mathrm{Tr}_{L/K} : L \times L \to K; (\alpha, \beta) \mapsto \mathrm{Tr}_{L/K}(\alpha\beta)$$

is a symmetric bilinear form over K. The extension is *separable* if the form is non-degenerate, in which case each basis $\{\alpha_1, ..., \alpha_n\}$ of L as a K-vector space has a dual basis $\{\beta_1, ..., \beta_n\}$ defined by the property:

$$\mathrm{Tr}_{L/K}(\alpha_i\beta_j) = \delta_{ij} := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

If $\mathrm{char}(K) = 0$ (i.e. $\mathbf{Z} \subset K$), then all extensions of K are separable, since $\mathrm{Tr}_{L/K}(\alpha\alpha^{-1}) = n = [L : K]$ is non-zero and then we say K is *perfect*. If $\mathrm{char}(K) = p$ (i.e. $\mathbf{Z}/p\mathbf{Z} \subset K$) then the form may be degenerate for extensions of degree divisible by p. Since all our fields are of characteristic zero, we will not worry about the characteristic p subtleties here.

If A is integrally closed in its field of fractions K, and $K \subset L$ is a finite field extension, then each $\alpha \in L$ that is integral over A has minimal polynomial $\lambda^d + a_{d-1}\lambda^{d-1} + ... + a_0$ with coefficients in A. That is because each root of the minimal polynomial (in its splitting field) is integral over A, and so the coefficients of the minimal polynomial, being symmetric polynomials in the roots, must be integral over A (by the proof of Corollary 9.2) and being in K, must therefore also be in A. This generalizes the same result in §8, which was more easily proved using Gauss' Lemma when A is a UFD.

Finite Generatedness of Integral Closure: If A is a Noetherian domain that is integrally closed in its field of fractions K, and if $K \subset L$ is a finite separable extension, then $\overline{A} \subset L$ is a finitely generated A-module.

Proof: Start with a basis $\{v_1, ..., v_n\}$ of L over K. By Remark (b) above, we know that $L = \overline{A}_S$ where $S = A - 0$, so we may multiply the v_i by elements $s_i \in A \subset K$ to obtain a basis $\{\alpha_1, ..., \alpha_n\}$ of L where each $\alpha_i \in \overline{A}$.

Let $\{\beta_1, ..., \beta_n\}$ be the dual basis. I claim that $\overline{A} \subset \beta_1 A + ... + \beta_n A$.

To see this, note that any $\alpha \in \overline{A}$ expands as $\alpha = \sum \gamma_j \beta_j$ for $\gamma_j \in K$, since the β_j are a basis, and therefore we can recover the γ_i coefficients as:

$$\mathrm{Tr}_{L/K}(\alpha\alpha_i) = \sum_j \mathrm{Tr}_{L/K}(\gamma_j \beta_j \alpha_i) = \gamma_i$$

But each $\alpha\alpha_i \in \overline{A}$, so $\gamma_i = \mathrm{Tr}_{L/K}(\alpha\alpha_i) \in A$, and the claim is proved.

Back to Proposition 9.3: The proposition now follows if X is normal. Otherwise, let $\mathbf{C}[y_1, ..., y_d] \subset \mathbf{C}[X]$ come from Noether normalization. Since $\mathbf{C}[X]$ is a finitely generated $\mathbf{C}[y_1, ..., y_d]$-module, it follows that the integral closures of $\mathbf{C}[y_1, ..., y_d]$ and of $\mathbf{C}[X]$ in K are the same.

Example: If X is any affine variety, then the affine variety Y satisfying:

$$\overline{\mathbf{C}[X]} = \mathbf{C}[Y] \subset \mathbf{C}(X)$$

comes with a birational finite map $\Phi : Y \to X$, the *canonical normalization* of X in its own field of fractions. The canonical normalization is the unique (up to isomorphism) birational finite map from a normal affine variety to X.

Next, we look for a local characterization of normality.

Definition: If X is any variety, then the *stalk* of $\mathcal{O}_X$ at $p \in X$ is the ring:

$$\mathcal{O}_{X,p} := \bigcup_{\{U|p\in U\}} \mathcal{O}_X(U) \subset \mathbf{C}(X)$$

consisting of all *germs* of rational functions defined at p.

Note: The stalk of germs of *analytic* functions at a point $p \in X$ of a complex manifold is the ring of convergent power series at $0 \in \mathbf{C}^n$, and is truly local in nature, saying nothing about the global geometry of X. The stalks $\mathcal{O}_{X,p}$, however, do retain information about the global geometry of a variety. For example, the field of fractions of $\mathcal{O}_{X,p}$ is $\mathbf{C}(X)$, the field of rational functions!

Observations: (a) If $V \subset X$ is an open set containing p, then $\mathcal{O}_{V,p} = \mathcal{O}_{X,p}$. This follows immediately from the fact that $\mathcal{O}_X(U) \subset \mathcal{O}_X(W)$ when $W \subset U$.

(b) Each ring $\mathcal{O}_{X,p}$ is a *local ring* with maximal ideal:

$$m_p := \{\phi \in \mathcal{O}_{X,p} \mid \phi(p) = 0\}$$

(since all other germs of rational functions in $\mathcal{O}_{X,p}$ are invertible)

(c) For any regular map $\Phi : X \to Y$ there are pull-backs of stalks:

$$\Phi^* : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p} \quad \text{with} \quad \Phi^*(m_q) \subseteq m_p$$

whenever $\Phi(p) = q$ (see Proposition 7.4 and the Remark following it).

(d) If $p \in Y$ and Y is affine, let $I(p) \subset \mathbf{C}[Y]$ be the maximal ideal. Then:

$$\mathcal{O}_{Y,p} = \mathbf{C}[Y]_{I(p)} \subset \mathbf{C}(Y)$$

since the regular functions on Y that can be denominators of germs of rational functions at p are precisely those that do not vanish at p. And if $\Phi : X \to Y$ is a regular map of affine varieties and $\Phi(p) = q$, then:

$$\Phi^*(m_q) = \Phi^*(I(q)\mathbf{C}[Y]_{I(q)}) = \Phi^*(I(q))\mathcal{O}_{X,p} \subset m_p$$

by the fundamental (localization) correspondence of §7.

(e) If $U \subset X$ is open, then (*by definition!*) $\mathcal{O}_X(U) = \bigcap_{\{p|p\in U\}} \mathcal{O}_{X,p}$ and in particular, if X is affine, then by Proposition 3.3,

$$\mathbf{C}[X] = \mathcal{O}_X(X) = \bigcap_{\{p|p\in X\}} \mathcal{O}_{X,p} = \bigcap_{p\in X} \mathbf{C}[X]_{I(p)}$$

and if $U = X - V(g) \subset X$ is a basic open affine, then $\mathbf{C}[X]_{\overline{g}} = \bigcap_{\{p|p\in U\}} \mathbf{C}[X]_{I(p)}$

Proposition 9.4: An affine variety X is normal if and only if each of the stalks $\mathcal{O}_{X,p}$ is integrally closed (in its field of fractions $\mathbf{C}(X)$).

Proof: Using Remark (b), it follows immediately that if X is normal, i.e. if $\mathbf{C}[X] = \overline{\mathbf{C}[X]}$, then every localization $\mathbf{C}[X]_S = \overline{\mathbf{C}[X]_S}$. In particular, each stalk $\mathcal{O}_{X,p} = \mathbf{C}[X]_{I(p)}$ is integrally closed.

For the converse, use Observation (e) and Proposition 3.3. If each $\mathcal{O}_{X,p}$ is integrally closed, then:

$$\overline{\mathbf{C}[X]} \subseteq \bigcap_{p\in X} \overline{\mathcal{O}_{X,p}} = \bigcap_{p\in X} \mathcal{O}_{X,p} = \mathbf{C}[X]$$

Definition: A quasi-projective variety X is *normal at* p if the stalk $\mathcal{O}_{X,p}$ is integrally closed (in $\mathbf{C}(X)$). X is *normal* if X is normal at all of its points.

Note: By Proposition 9.4, the two notions of normal agree for affine varieties.

Example: If X has an open cover by open subsets of $\mathbf{C}^n$, then X is normal.

Proposition 9.5: In *every* quasi-projective variety X, the subset:

$$\text{Norm}(X) := \{q \in X \mid X \text{ is normal at } q\} \subset X$$

is open and dense (i.e. not empty).

Proof: If $X = \cup U_i$ is a open cover by affine varieties and the Proposition holds for each U_i, then it holds for X. So we may assume that X is affine.

Consider the normalization map $\Phi : Y \to X$. Since Φ is birational, we know from Proposition 8.5 that there is an open subset $U \subset X$ such that $\Phi : \Phi^{-1}(U) \to U$ is an isomorphism. But then $\Phi^* : \mathcal{O}_{X,q} \to \mathcal{O}_{Y,\Phi^{-1}(q)}$ is an isomorphism of stalks at every $q \in U$, so $U \subseteq \text{Norm}(X)$.

A little care shows that Norm(X) is itself open. If $q \in \text{Norm}(X)$, then $\mathbf{C}[Y] = \overline{\mathbf{C}[X]} \subset \overline{\mathbf{C}[X]_{I(q)}} = \mathbf{C}[X]_{I(q)}$ so any generators $\phi_1, ..., \phi_m \in \mathbf{C}[Y]$ as a $\mathbf{C}[X]$-module can be written as $\phi_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbf{C}[X]$ and $b_i(q) \neq 0$. Let $f = \prod b_i$. Then $q \in X - V(f)$ and $\mathbf{C}[Y]_f \subset \mathbf{C}[X]_f$, hence $\overline{\mathbf{C}[X]_f} = \mathbf{C}[X]_f$, and so the basic open set $X - V(f)$ is normal and contained in Norm(X). Since this is true at every point of Norm(X), we see that Norm(X) is open.

Next, I want to normalize an arbitrary quasi-projective variety Y in an arbitrary finite extension $\mathbf{C}(Y) \subset K$ of its field of fractions.

The Construction: For each point $q \in Y$, let

$$S_q = \{\text{maximal ideals in } \overline{\mathcal{O}_{Y,q}} \subset K\}$$

and then let

$$\Phi : X = \coprod_{q \in Y} S_q \to Y;\ p \mapsto q \Leftrightarrow p \in S_q$$

(this defines the normalization *as a set* mapping to Y).

The Topology: If $q \in U \subset Y$ is an affine neighborhood, let $\Phi_U : V \to U$ be the normalization in K coming from Proposition 9.3, with $\mathbf{C}[V] = \overline{\mathbf{C}[U]} \subset K$. I claim there is a natural bijection: $\Phi_U^{-1}(q) \leftrightarrow S_q$. Indeed, we have:

$$\begin{aligned}\Phi_U^{-1}(q) &\leftrightarrow \{\text{maximal ideals in } \overline{\mathbf{C}[U]} \text{ containing } I(q)\} \text{ (Nullstellensatz)} \\ &\leftrightarrow \{\text{maximal ideals in } \overline{\mathbf{C}[U]}_S \text{ for } S = \mathbf{C}[U] - I(q)\} \text{ (Going Up)} \\ &\leftrightarrow \{\text{maximal ideals in } \overline{\mathbf{C}[U]_{I(q)}}\} = S_q \text{ (Localizing Integral Closures)}\end{aligned}$$

Notice that this tells us each S_q is a *finite* set (Proposition 7.5) and allows us to identify $V = \Phi^{-1}(U) \subset X$ as a subset of X, which we declare to be open (making the map Φ continuous). We give X the topology generated by the open subsets of all such sets V (so the inclusions $V \subset X$ are continuous).

The Sheaf: If $W \subset X$ is an open set, define:

$$\mathcal{O}_X(W) = \{\phi \in K \mid \phi \in \left(\overline{\mathcal{O}_{Y,\Phi(w)}}\right)_{m_w} \quad \text{for all} \quad w \in W\}$$

where $m_w \subset \overline{\mathcal{O}_{Y,\Phi(w)}}$ is the maximal ideal corresponding to $w \in S_{\Phi(w)}$. In other words, the stalks of $\mathcal{O}_X$ are the local rings $\mathcal{O}_{X,x} := \left(\overline{\mathcal{O}_{Y,\Phi(x)}}\right)_{m_x} \subset K$. And if $x \in V$ for some $U \subset Y$ affine and open and $V = \Phi^{-1}(U)$, then

$$\mathcal{O}_{V,x} = \overline{\mathbf{C}[U]}_{I(x)} = \left(\overline{\mathbf{C}[U]_{I(\Phi(x))}}\right)_{m_x} \subset K$$

so the stalks are the same, and it follows that all the inclusions $V \subset X$ are regular maps, *if X is isomorphic to a quasi-projective variety.* For now, we only know that X is covered by affine varieties, not that X is quasi-projective. On the other hand, it is easy to see that X is the "universal" normalization, in the sense that a quasi-projective normalization $\Psi : X' \to Y$, *if it exists*, must be isomorphic to this particular X, hence any two such are isomorphic to each other!

Proposition 9.6: The normalization of a *projective* variety Y in any finite field extension $\mathbf{C}(Y) \subset K$ is a *projective* variety.

Proof: Let $Y \subset \mathbf{CP}^n$, with homogeneous ring $\mathbf{C}[Y] = \mathbf{C}[y_0, ..., y_n]/P$. Let $y = \sum a_i \overline{y}_i \in \mathbf{C}[Y]_1$ be any non-zero element, and consider:

$$\mathbf{C}[Y] \subset \mathbf{C}(Y)[y] \subset \mathbf{C}(Y)(y) = (\text{the field of fractions of } \mathbf{C}[Y])$$

It is easy to see that the integral closure of $\mathbf{C}[Y]$ in $K(y)$:

$$R := \overline{\mathbf{C}[Y]} \subset K[y] \subset K(y)$$

(recall that $K[y]$ is integrally closed in $K(y)$) is a graded ring:

$$R = \bigoplus_{d=0}^{\infty} R_d; \;\; R_d = R \cap K[y]_d$$

with constants $R_0 = \mathbf{C}$. We know R is finitely generated as a $\mathbf{C}[Y]$-module (finite generatedness!), so in particular, there are generators $z_0, ..., z_m \in K[y]$ with $R = \mathbf{C}[z_0, ..., z_m]/Q$ and we would like to argue (as in the affine case) that $R = \mathbf{C}[X]$ for the projective normalization $\Phi : X \to Y$.

This line of reasoning is basically correct, but needs more work because *it may not be possible to choose the z_i generators to all have degree* 1. The Proposition below will tell us that this will be possible, however, if we replace $Y \subset \mathbf{CP}^n$ by a suitable Segre reembedding $Y = Y_{n,d} \subset \mathbf{CP}^{\binom{n}{d}-1}$ (see §5).

Proposition 9.7: Suppose that

$$R = \mathbf{C}[z_0, ..., z_m]/I = \bigoplus_{d=0}^{\infty} R_d$$

is a weighted homogeneous $\mathbf{C}$-algebra, with generators z_i of degrees $d_i > 0$. Then there is a degree d such that the sub $\mathbf{C}$-algebra:

$$R^{(d)} := \bigoplus_{k=0}^{\infty} R_{dk} \subset R$$

is generated by elements of degree 1 in $R^{(d)}$ (= degree d in R).

Proof: Let $n = \text{lcm}(\{d_i\})$ so $n = d_i e_i$ for each d_i and integers e_i. We claim that $d = mn$ will satisfy the Proposition. To see this, suppose $\prod_{i=0}^m z_i^{f_i}$ is a monomial of degree $\sum d_i f_i = dk$ for some $k > 0$. If $k = 1$, there is nothing to do. If $k > 1$, then some $f_i \geq e_i$, and we can pull out a factor of $z_i^{e_i}$ (of degree n), and we can keep doing this until the left-over monomial has degree exactly d. Then we can group together the monomials we have pulled out to express the original $\prod_{i=0}^m z_i^{f_i}$ as a product of monomials of degree d.

Example: Consider the graded domain:

$$R = \mathbf{C}[z_0, z_1, z_2]/\langle z_0^2 - z_1^2 - z_2^3 \rangle \text{ where } d_0 = d_1 = 3, d_2 = 2$$

Then according to the Proposition, we can take $d = 12$. The monomials:

$$z_0^4, z_0^3 z_1, z_0^2 z_1^2, z_0 z_1^3, z_1^4, z_0^2 z_2^3, z_0 z_1 z_2^3, z_1^2 z_2^3, z_2^6$$

are all the monomials of degree 12, and then using the relation, we only need $z_0 z_1^3, z_1^4, z_0 z_1 z_2^3, z_1^2 z_2^3, z_2^6$ to span R_{12}, meaning that we get:

$$R^{(12)} \cong \mathbf{C}[x_0, ..., x_4]/P = \mathbf{C}[X]$$

from the Proposition, where $X = V(P) \subset \mathbf{CP}^4$. (What are the equations?)

For example, following the proof, we rewrite the following monomial:

$$z_0^3 z_1^3 z_2^3 = (z_0^2)(z_0 z_1^3 z_2^3) = (z_0^2)(z_1^2)(z_0 z_1 z_2^3) = (z_0^2 z_1^2)(z_0 z_1 z_2^3)$$

as a product of two monomials of degree 12.

Back to the Proof of Proposition 9.6: Apply Proposition 9.7 to the ring $R = \mathbf{C}[z_0, ..., z_m]/Q \subset K[y]$ to get the new ring:

$$R^{(d)} = \mathbf{C}[x_0, ..., x_l]/Q' = \mathbf{C}[X] \text{ for } X = V(Q') \subset \mathbf{CP}^l$$

This no longer contains $\mathbf{C}[Y]$, of course, but it does contain the homogeneous coordinate ring of the Segre re-embedding $Y = Y_{n,d} \subset \mathbf{CP}^{\binom{n}{d}-1}$:

$$\mathbf{C}[Y]^{(d)} = \mathbf{C}[Y_{n,d}] \subset \mathbf{C}(Y_{n,d})[y^d] \subset (\text{the field of fractions of } \mathbf{C}[Y_{n,d}])$$

by Exercise 5.?. But now a moment's reflection will convince you that $R^{(d)}$ is the integral closure of $\mathbf{C}[Y_{n,d}] = \mathbf{C}[Y]^{(d)}$ in $K[y^d] \subset K(y^d)$, and then Exercise 7.? gives a finite map $\Phi : X \to Y_{n,d} = Y$.

X is a normal projective variety because $\mathbf{C}[X]$ is integrally closed in its field of fractions $K(y^d)$. To see why this implies normality, consider the affine open sets $U_i = Y - V(y_i)$ of Proposition 4.7. Then $\mathbf{C}[U_0] = \mathbf{C}[1, \frac{x_1}{x_0}, ..., \frac{x_l}{x_0}]/d_0(P') = \mathbf{C}[X]_{x_0} \cap K[y^d]_0 \subset K = K[y^d]_0$, and so since $\mathbf{C}[X]$ is integrally closed, it follows that $\mathbf{C}[X]_{x_0}$ is integrally closed, and so too is its degree zero part (again after a moment's reflection). Thus $\mathbf{C}[U_0]$ and, likewise, each $\mathbf{C}[U_i]$ is integrally closed in its field of fractions K.

So we have a normal variety X mapping finitely to Y with $\mathbf{C}(X) = K$, the given extension of $\mathbf{C}(Y)$. This tells us precisely that the map $\Phi : X \to Y$ is the normalization of Y for the field extension $\mathbf{C}(Y) \subset K$, as desired.

Corollary 9.8: The normalization of a quasi-projective variety Y in any finite extension $\mathbf{C}(Y) \subset K$ is again a quasi-projective variety.

Proof: Normalize the closure $Y \subset \overline{Y} \subset \mathbf{CP}^n$ for any embedding in $\mathbf{CP}^n$ by Proposition 9.6 to get $\Phi : \overline{X} \to \overline{Y}$, and let $X = \Phi^{-1}(Y) \subset \overline{X}$. Then the restricted finite map $\Phi|_X : X \to Y$ is the normalization of Y by a quasi-projective variety X.

Zariski's Main Theorem: If Y is a normal variety and $\Phi : X \to Y$ is any birational map with finite fibers, then there is an open subset $U \subset Y$ such that Φ is an isomorphism from X to U. In particular, Φ is injective(!)

Remark: This is completely false when Y is not normal. We've already seen one example of this, with the birational map:

$$\Phi : \mathbf{C}^1 \to V(y^2 - x^3) \subset \mathbf{C}^2;\ t \mapsto (t^2, t^3)$$

which is a birational homeomorphism but not an isomorphism.

For another example, consider the map:

$$\Phi : \mathbf{C}^1 \to V(y^2 - x^2(x+1)) \subset \mathbf{C}^2;\ t \mapsto (t^2 - 1, t(t^2 - 1))$$

This is birational but $\Phi^{-1}(0,0) = \{\pm 1\}$ consists of two points. This sort of behavior cannot occur when the target is normal!

Proof of the Main Theorem: This is another great application of Grothendieck's Theorem (§7). By that theorem, Φ extends to a *finite* map $\Phi' : X' \to Y$ and $X \subset X'$ is open. But a finite birational map to a normal variety is an isomorphism(!), as can be checked on an affine open cover.