Math 6130 Notes. Fall 2002.

10. Non-singular Varieties. In §9 we produced a canonical normalization map $\Phi : X \to Y$ given a variety Y and a finite field extension $\mathbf{C}(Y) \subset K$. If we forget about Y and only consider the field K, then we can ask for a projective variety X with $\mathbf{C}(X) = K$ with better properties than normality. Non-singularity is such a property, which implies that X is, in paricular, a complex analytic manifold.

Definition: The *Zariski cotangent space* of a variety X at a point $p \in X$ is the vector space:

$$m_p/m_p^2$$

(recall that $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal in the stalk at p).

Basic Remarks: (a) Cotangent spaces are finite-dimensional, since m_p/m_p^2 is generated (as a vector space) by any set of generators of m_p (as an ideal).

(b) Cotangent spaces pull back under regular maps. Given $\Phi : X \to Y$ with $\Phi(p) = q$, then $\Phi^* : m_q \to m_p$ induces $d\Phi : m_q/m_q^2 \to m_p/m_p^2$.

(c) Cotangent spaces can be expressed in terms of coordinate rings of affine varieties. If $p \in U \subset X$ is any affine neighborhood and $I(p) \subset \mathbf{C}[U]$ is the (maximal) ideal of p, then the natural map:

$$I(p)/I(p)^2 \to m_p/m_p^2$$

is an isomorphism of vector spaces. Injectivity is obvious from the injectivity of $I(p) \hookrightarrow m_p \subset \mathbf{C}[U]_{I(p)}$. For surjectivity, notice that if $\frac{f}{s} \in m_p$, then

$$\frac{f}{s} - \frac{f \cdot s(p)^{-1}}{1} = \frac{f(1 - s \cdot s(p)^{-1})}{s} \in m_p^2$$

(d) If $X \subseteq \mathbf{C}^n$ and $I(X) = \langle f_1, ..., f_m \rangle$, then:

$$m_p/m_p^2 \cong I(p)/I(p)^2 \cong \bigoplus_{i=1}^{n} \mathbf{C}[x_i - p_i] / \sum_{j=1}^{m} \mathbf{C}\left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p)[x_i - p_i]\right)$$

since the $x_i - p_i$ generate $I(p)$, and the linear combinations of the $x_i - p_i$ that belong to $I(p)^2$ are precisely the first-order terms in the Taylor expansions of polynomials $f \in I(X)$. But if $f \in I(X)$, then the first-order term in its Taylor expansion is a linear combination of the first-order terms in the Taylor expansions of the generators of $I(X)$.

Example: If $X \subset \mathbf{C}^n$ is an irreducible hypersurface with $I(X) = \langle f \rangle$, then:

$$m_p/m_p^2 \cong \mathbf{C}^{n-1} \text{ or } \mathbf{C}^n$$

and the latter only occurs on the closed subset where the gradient vanishes:

$$f(p) = 0, \nabla f(p) = (0, ..., 0) \Leftrightarrow p \in V(\langle f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \rangle)$$

Notice that the gradient cannot vanish identically on $V(f)$, since f is an irreducible polynomial of positive degree. So there is an open, dense subset $U \subset V(f)$ where the dimension of the cotangent space is $n - 1$.

Proposition 10.1: For any variety X, the function $e : X \to \mathbf{Z}$:

$$e(p) = \dim \left(m_p/m_p^2 \right)$$

is upper-semi-continuous (see §7) and the minimum value $e(p) = \dim(X)$ is taken on a dense open subset $U \subset X$.

Proof: We may assume that $X \subset \mathbf{C}^n$ is affine, with $I(X) = \langle f_1, ..., f_m \rangle$, since upper-semi-continuity can be checked on the open sets of an open cover, and then it follows from linear algebra and Basic Remark (d) above that $e(p)$ is the dimension of the kernel of the "Jacobian matrix:"

$$J(x_1, ..., x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

which is upper-semicontinuous, since:

$$X_{n-a+1} := \{p \in X \ | e(p) \geq n - a + 1\} = X \cap V(\{J_{IK}(x_1, ..., x_n)\})$$

where J_{IK} ranges over all determinants of $a \times a$ minors of the Jacobain matrix. Upper-semi-continuity implies that the minimum value of $e(p)$ is attained on an open subset of X. We only need to see that this value is equal to $\dim(X)$. For this we use:

Field Theory IV. Every finite, separable extension of fields $K \subset L$ has a *primitive element*, i.e. there is an $\alpha \in L$ such that $L = K[\alpha]$.

If $d = \dim(X)$, take the subring $\mathbf{C}[y_1, ..., y_d] \subset \mathbf{C}[X]$ given by Noether Normalization and consider the finite field extension $\mathbf{C}(y_1, ..., y_d) \subset \mathbf{C}(X)$. If $\alpha \in \mathbf{C}(X)$ is primitive, then $\mathbf{C}(y_1, ..., y_d)[y]/g(y) \xrightarrow{\sim} \mathbf{C}(X);\ y \mapsto \alpha$ for some $g \in \mathbf{C}(y_1, ..., y_d)[y_{d+1}]$, and we may assume the coefficients of $g(y)$ are polynomials in $y_1, ..., y_d$ (clearing denominators) and $g(y_1, ..., y_d, y)$ is irreducible. Thus if $V(g) \subset \mathbf{C}^{d+1}$, then $\mathbf{C}(X) \cong \mathbf{C}(V(g))$ so by Proposition 7.5 there is a birational map:

$$\Phi : X -\!\!> V(g) \subset \mathbf{C}^{d+1}$$

and by Proposition 8.5, $\Phi : \Phi^{-1}(U) \to U$ is an isomorphism for some open subset $U \subset V(g)$.

But this means that for all points $p \in \Phi^{-1}(U)$, the dimension of the cotangent space at $p \in X$ is the same as the dimension of the cotangent space at $q = \Phi(p) \in V(g)$. We've already seen from the Example above that $\dim(m_q/m_q^2) = d$ on an open subset of $V(g)$, so this dimension is also attained on an open subset of X, completing the proof of the Proposition.

Definition: (a) $p \in X$ is a *singular point* of X if $\dim(m_p/m_p^2) > \dim(X)$. Otherwise $p \in X$ is a *non-singular point* of X.

(b) The variety X is *non-singular* if every $p \in X$ is a non-singular point.

In the affine case, a point $p \in X \subset \mathbf{C}^n$ with $I(X) = \langle f_1, ..., f_m \rangle$ is non-singular if and only if the Jacobian matrix has rank $n - d$ rank at p. But in this case there is a subset: $\{f_{j_1}, ..., f_{j_{n-d}}\} \subset \{f_1, ..., f_m\}$ such that the first-order terms in the Taylor series span the same space:

$$\sum_{k=1}^{n-d} \mathbf{C} \left(\sum_{i=1}^{n} \frac{\partial f_{j_k}}{\partial x_i}(p)[(x_i - p_i)] \right) = \sum_{j=1}^{m} \mathbf{C} \left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p)[x_i - p_i] \right)$$

But now recall the *implicit function theorem* from analysis, which says that in this case there are *analytic* local coordinates (convergent power series): $z_1 = z_1(x_1, ..., x_n), ..., z_d = z_d(x_1, ..., x_n)$ at p meaning that for some analytic neighborhood $p \in U$, the map:

$$(z_1, ..., z_d) : U \to \mathbf{C}^d$$

is an isomorphism with a neighborhood of the origin in $\mathbf{C}^d$. In other words, a variety has analytic local coordinates at every non-singular point, and a non-singular variety is thus a complex analytic manifold. The best we can do with *regular* functions and our Zariski topology is the following:

Proposition 10.2: If $p \in X$ is an arbitrary point and $g_1, ..., g_d \in m_p$, then:

$$\langle g_1, ..., g_d \rangle = m_p \;\Leftrightarrow\; \overline{g}_1, ..., \overline{g}_d \text{ span } m_p/m_p^2$$

(this is the converse to Basic Remark (a) above!)

Proof: Consider the map of finitely generated $\mathcal{O}_{X,p}$-modules:

$$\phi : \bigoplus_{i=1}^{d} \mathcal{O}_{X,p} \to m_p;\ (f_1, ..., f_d) \mapsto \sum_{i=1}^{d} f_i g_i$$

Then ϕ is surjective if and only if $\langle g_1, ..., g_d \rangle = m_p$, and ϕ induces a surjective map of vector spaces

$$\overline{\phi} : \mathbf{C}^d = \bigoplus_{i=1}^{d} \mathcal{O}_{X,p}/m_p \to m_p/m_p^2;\ (a_1, ..., a_d) \mapsto \sum_{i=1}^{d} a_i \overline{g}_i \in m_p/m_p^2$$

if and only if the $\overline{g}_1, ..., \overline{g}_d$ span m_p/m_p^2. The Proposition now follows from:

Nakayama's Lemma II: If M and N are finitely generated modules over a local Noetherian ring A, and:

$$\phi : M \to N$$

is an A-module homomorphism, then ϕ is surjective if and only if:

$$\overline{\phi} : M/mM \to N/mN$$

is a surjective map of vector spaces over the "residue" field A/mA.

Proof: We want to show that $Q = 0$, for the cokernel $Q = N/\phi(M)$. The surjectivity of $\overline{\phi}$ implies $Q = mQ$ (since Q/mQ is the cokernel of $\overline{\phi}$) so we can apply Nakayama I (§6) to Q to conclude that there is an element:

$$a = 1 + b \in A \text{ with } b \in m \text{ such that } aQ = 0$$

But since A is a local ring, such an $a \in A$ is a *unit*, so $Q = 0$, as desired!

Definition: If $p \in X$ is a non-singular point, then any $g_1, ..., g_d \in m_p$ that lift a basis of the cotangent space m_p/m_p^2 are called a *local system of parameters*.

Note: By the Proposition, a local system of parameters generates m_p, but the g_i are almost never the germs of a set of *local coordinates* of X. That is, if $p \in U$ is an affine neighborhood on which each of the $g_1, ..., g_d$ are defined, then the regular map:

$$\Phi : U \to \mathbf{C}^d;\ \Phi(p) = (g_1(p), ..., g_d(p))$$

is almost never an isomorphism with an open subset of $\mathbf{C}^d$. Instead, for small enough neighborhoods of p, the map is *étale*, which means, roughly, that it is (an open subset of) a topological *covering space* of an open subset of $\mathbf{C}^d$.

Example: Take $p = (0,0) \in V(y^2 - x(x-1)(x-\lambda))$, the affine elliptic curve. Then the Zariski cotangent space is:

$$m_p/m_p^2 \cong \mathbf{C}[x] \oplus \mathbf{C}[y]/\mathbf{C}(-\lambda[x]) \cong \mathbf{C}$$

so p is a non-singular point, and $y \in m_p$ is a local parameter, but x is not (indeed $x = uy^2$ where $u = (x-1)^{-1}(x-\lambda)^{-1} \in m_p$). Under the map:

$$\Phi : V(y^2 - x(x-1)(x-\lambda)) \to \mathbf{C};\ (x,y) \mapsto y$$

there is a neighborhood of p (in the Zariski topology) that is a $3:1$ cover of $\mathbf{C}^1$ minus two points. Since the only open subsets of the elliptic curve in the Zariski topology are the complements of finite sets, the map stays $3:1$ over most of any neighborhood of p, no matter how "small" it is!

Here is an extremely important property of nonsingular points:

UFD Theorem 10.3: The stalk $\mathcal{O}_{X,p}$ of a quasi-projective variety X at any non-singular point $p \in X$ is always a UFD.

This will require our longest excursion yet into commutative algebra. Before we do this, recall that a UFD is integrally closed (§9), so:

Corollary 10.4: A non-singular variety is a normal variety.

But not vice versa, in general! (unless $\dim(X) = 1$....more on that later)

Example: The point $p = (0,0,0) \in V(y^2 - xz) \subset \mathbf{C}^3$ is singular, since the gradient of $y^2 - xz$ vanishes at the origin. On the other hand,

$$\mathbf{C}[x,y,z]/\langle y^2 - xz\rangle$$

is integrally closed.

Completions: Let A be a local Noetherian domain with maximal ideal m. The *completion* of (A, m) is the inverse limit ring:

$$\widehat{A} := \lim_{\leftarrow} A/m^d$$

consisting of all *inverse systems*:

$$\{(..., b_2, b_1, b_0) \mid b_d \in A/m^{d+1} \text{ and } \bar{b}_{d+1} = b_d \in A/m^{d+1}\}$$

To get a feel for this, think of A as a topological space with nested subsets:

$$... \subset m^3 \subset m^2 \subset m \subset A$$

and their translates $a + m^d$ as a basis of open sets for the "m-adic topology." The analogy with the ordinary topology on $\mathbf{R}^n$ is not perfect, since here "close" is a transitive notion (i.e. $a - a' \in m^d$ and $a' - a'' \in m^d \Rightarrow a - a'' \in m^d$) which is certainly not the case in $\mathbf{R}^n$, and explains why we can consider inverse systems rather than Cauchy sequences of elements of A. The first result we need says that no nonzero element of A is arbitrarily close to zero.

Krull's Theorem: In this setting (local Noetherian domain (A, m))

$$\bigcap_{d=0}^{\infty} m^d = 0$$

Proof: Suppose $a_1, ..., a_n \in A$ generate m. Then an element $a \in \cap m^d$ is a homogeneous polynomial $F_d(a_1, ..., a_n)$ of degree d in the a_i, for every d.

Let $I \subset A[x_1, ..., x_n]$ be the smallest ideal containing all $F_d(x_1, ..., x_n)$, thought of as homogeneous polynomials in the variables $x_1, ..., x_n$. Since $A[x_1, ..., x_n]$ is Noetherian (Proposition 1.1!) we know that I only requires finitely many generators, say $I = \langle F_1(x_1, ..., x_n), ..., F_l(x_1, ..., x_n)\rangle$ so:

$$F_{l+1}(x_1, ..., x_n) = \sum_{i=1}^{l} F_{l+1-i}(x_1, ..., x_n) G_i(x_1, ..., x_n)$$

and then evaluating at $x_1 = a_1, ..., x_n = a_n$ gives:

$$a = \sum_{i=1}^{l} a G_i(a_1, ..., a_n) \text{ or } a(1 - \sum_{i=1}^{l} b_i) = 0 \text{ for } b_i = G_i(a_1, ..., a_n)$$

But A is a local ring and each $b_i \in m$, so $1 - \sum_{i=1}^{l} b_i$ is a unit, and $a = 0$.

Notice that when $A = \mathcal{O}_{X,p}$, Krull's theorem says that the only germ of a regular function which is "zero to all orders" at $p \in X$ is the zero germ, which is, of course, also the case with analytic (but not differentiable!) functions.

Corollary 1: The m-adic topology on A is Hausdorff.

Proof: Suppose $a \neq a' \in A$. Then by Krull's theorem, $a - a' \notin m^d$ for some $d > 0$. But then $(a + m^d) \cap (a' + m^d) = \emptyset$ (transitivity of closeness). Thus A is Hausdorff!

Corollary 2: The natural ring homomorphism:

$$\iota : A \to \widehat{A}; \quad a \mapsto (\overline{a}, \overline{a}, \overline{a}, ...)$$

is injective since an element of the kernel would be in every m^d.

And now we can say $\widehat{A}$ is the completion of A:

Corollary 2: $\widehat{A}$ topologically completes $A \subset \widehat{A}$.

Proof: Given a Cauchy sequence $a_1, a_2, a_3, ... \in A$, consider:

$$(..., b_2, b_1, b_0) \text{ with } b_d = \overline{a}_n \in A/m^{d+1} \text{ for all sufficiently large } n$$

The point is that Cauchy means that for any d, eventually:

$$a_{n+1} - a_n \in m^{d+1}$$

so this definition of the inverse limit is well-defined. If two Cauchy sequences $\{a_i\}$ and $\{a_i'\}$ have the same limit, i.e. if for any d, eventually:

$$a_n - a_n' \in m^{d+1}$$

then they give the same inverse limit, and conversely, given an inverse limit: $(..., b_2, b_1, b_0)$, we can lift arbitrarily to:

$$a_0, a_1, a_2, ... \in A \text{ such that } \overline{a}_d = b_d \in A/m^{d+1}$$

and this evidently Cauchy. So $\widehat{A}$ completes $A \subset \widehat{A}$.

Example: If A is a (not local) Noetherian ring and m is maximal then:

$$A/m^d \to A_S/m_S^d; \quad \overline{a} \to \overline{\left(\frac{a}{1}\right)}$$

is an isomorphism (for $S = A - m$) since any $f \in S$ is a unit in A/m^d.

Thus elements of $\widehat{A}_S$ can be written $(..., b_2, b_1, b_0)$ where $b_d \in A/m^{d+1}$

(a) For $m = \langle p \rangle \subset \mathbf{Z}$, the ring $\widehat{\mathbf{Z}}_S$ is the ring of p-adic integers:

$$\{(..., b_2, b_1, b_0) \mid b_d \in \mathbf{Z}/p^{d+1}\mathbf{Z} \text{ and } \overline{b_{d+1}} = b_d\}$$

But we can identify each b_d with an integer from 0 to $p^d - 1$:

$$b_d = c_0 + c_1 p + c_2 p^2 + ... + c_{d-1} p^{d-1}; \quad c_i \in \{0, ..., p-1\}$$

and this gives the more familiar description of the p-adic integers:

$$\mathbf{Z}_p = \{\sum_{d=0}^{\infty} c_d p^d\}$$

(b) Similarly, if $p = (p_1, ..., p_n) \in \mathbf{C}^n$ and $S = I(p)$, then:

$$\mathbf{C}\widehat{[x_1, ..., x_n]}_S \cong \mathbf{C}[[x - p_1, ..., x - p_n]]$$

is the ring of (formal) Taylor series at the point p.

Remarks: Ideals and modules are completed in the same way, and:

- If (A, m) is a Noetherian local ring, then so is $(\widehat{A}, \widehat{m})$.
- the natural map $\widehat{A} \otimes_A M \to \widehat{M}$ is an isomorphism when M is a finitely generated A-module and A is Noetherian.
- $\widehat{\widehat{M}} = \widehat{M}$ (i.e. completions are complete)
- If $M' \to M \to M''$ is an exact sequence of finitely generated A-modules, then the sequence of completions $\widehat{M}' \to \widehat{M} \to \widehat{M}''$ is also an exact sequence of finitely generated $\widehat{A}$-modules.

These remarks are not meant to be obvious. They are, however, standard results in commutative algebra. See, for example Atiyah-Macdonald's book *Introduction to Commutative Algebra* Chapter 10 for proofs. Even though the results are important, I don't want to duplicate too much standard material.

Let's return to $p \in X$ a non-singular point of a quasi-projective variety.

Strategy for the Proof of UFD Theorem 10.3: Let $A = \mathcal{O}_{X,p}$.

Step 1: Prove that if $g_1, ..., g_n \in \mathcal{O}_{X,p}$ is a local system of parameters, then the natural "evaluation" ring homomorphism:

$$ev : \mathbf{C}[[x_1, ..., x_n]] \to \widehat{\mathcal{O}}_{X,p}; \quad x_i \mapsto g_i$$

is an isomorphism (vastly generalizing Example (b) above!).

Step 2: Prove that the power series rings $\mathbf{C}[[x_1, ..., x_n]]$ are UFD's.

Step 3: Prove that if $\widehat{A}$ is a UFD, then $A \subset \widehat{A}$ is a UFD.

Step 1: A power series:

$$\sum_{d=0}^{\infty} F_d(x) \in \mathbf{C}[[x_1, ..., x_n]]$$

maps under the evaluation homorphism to the inverse limit:

$$(..., \overline{F_2(g) + F_1(g) + F_0(g)}, \overline{F_1(g) + F_0(g)}, \overline{F_0(g)}) \in \widehat{\mathcal{O}}_{X,p}$$

(where the $F_d(x)$ are homogeneous polynomials of degree d).

Since the $g_1, ..., g_n$ generate m_p, an arbitrary element $f \in m_p^d$ looks like:

$$f = \sum_{i_1+...+i_n=d+1} h_I g_1^{i_1} \cdots g_n^{i_n}; \quad h_I \in \mathcal{O}_{X,p}$$

and then if $F_d(x_1, ..., x_n) = \sum h_I(p) x_1^{i_1} \cdots x_n^{i_n}$ it follows that $F_d(g) - f \in m_p^{d+1}$. In other words, the map:

$$\mathbf{C}[x_1, ..., x_n] \to m_p^d/m_p^{d+1}; \; F_d(x) \to \overline{F_d(g)}$$

is surjective. But then it follows that ev it surjective since if, inductively, $\overline{F_{d-1}(g) + ... + F_0(g)} = b_{d-1}$ and $\overline{b}_d = b_{d-1} \in \mathcal{O}_{X,p}/m_p^d$, choose $F_d(x)$ so that:

$$\overline{F}_d(g) = b_d - \overline{F_{d-1}(g) - ... - F_0(g)} \in m_p^d/m_p^{d+1}$$

and then $\overline{F_d(g) + ... + F_0(g)} = b_d \in \mathcal{O}_{X,p}/m_p^{d+1}$, as desired.

For injectivity we need $F_d(g_1, ..., g_n) \in m^{d+1} \quad \Leftrightarrow F_d(x_1, ..., x_n) = 0$. But if $F_d(g) \in m^{d+1}$, then $F_d(g) = \sum_{|I|=d+1} h_I g_{i_1} \cdots g_{i_n} = \Phi_{d+1}(g)$ for some $\Phi_{d+1}(x) \in \mathcal{O}_{X,p}[x_1, ..., x_n]_{d+1}$ and then (as in the proof of Noether normalization in §1) there are constants $c_1, ..., c_{n-1}$ (after possibly reordering the x_i) and a nonzero(!) additional constant $c \in \mathbf{C}$ so that, setting $y_1 = x_1 + c_1 x_n, ..., y_{n-1} = x_{n-1} + c_{n-1} x_n$ gives:

$$F_d(x_1, ..., x_n) = a x_n^d + G_1(y) x_n^{d-1} + ... + G_d(y),$$

for $G_i(y) \in \mathbf{C}[y_1, ..., y_{n-1}]_i$ and then for some $\Gamma_i(y) \in \mathcal{O}_{X,p}[y_1, ..., y_{n-1}]_i$:

$$\Phi_{d+1}(x_1, ..., x_n) = \Gamma_0(y) x_n^{d+1} + \Gamma_1(y) x_n^d + ... + \Gamma_{d+1}(y)$$

so evaluating at $g_1, ..., g_n$ and subtracting gives $(c - \Gamma_0 g_n) g_n^d \in \langle f_1, ..., f_{n-1} \rangle$ where $f_i = g_i + c_i g_n$. But notice that $c - \Gamma_0 g_n$ is a unit in $\mathcal{O}_{X,p}$, so in fact $g_n^d \in \langle f_1, ..., f_{n-1} \rangle$. I claim that this is impossible when we assume that the $g_1, ..., g_n$ (and hence also $f_1, ..., f_{n-1}, g_n$) is a local system of parameters.

Here's the argument. If $g_n^d \in \langle f_1, ..., f_{n-1} \rangle$ and $\langle f_1, ..., f_{n-1}, g_n \rangle = m_p$, then there is an affine neighborhood U of $p \in X$ such that:

- $f_1, ..., f_{n-1}, g_n \in \mathbf{C}[U]$,
- $\langle f_1, ..., f_{n-1}, g_n \rangle = I(p) \subset \mathbf{C}[U]$, and
- $g_n^d \in \langle f_1, ..., f_{n-1} \rangle \subset \mathbf{C}[U]$

(each successive item above may require shrinking the neighborhood!)

Then $\mathbf{C}[U]/\langle f_1, ..., f_{n-1} \rangle$ is a vector space over $\mathbf{C} = \mathbf{C}[U]/\langle f_1, ..., f_{n-1}, g_n \rangle$ of dimension d (or less) so it contains only finitely many maximal ideals. But this implies that the irreducible components of $V(\langle f_1, ..., f_{n-1} \rangle) \subset U$ are *points*, which is impossible if $\dim(U) = n$, since by Krull's principal ideal theorem every component of $V(\langle f_1, ..., f_{n-1} \rangle)$ has dimension 1 or more!

Example: Step 1 says in particular that regular functions at p have (unique) Taylor series expansions. For example, let $X = V(y^2 - (x^3 - 1))$ and $p = (1, 0)$. Then:

$$y \in m_p \text{ is a local parameter}$$

and the Taylor series expansion of x in terms of y is $ev^{-1}(x)$ for:

$$ev : \mathbf{C}[[t]] \to \widehat{\mathcal{O}}_{X,p}$$

Since on X we have the (non-algebraic) relation $x = \sqrt[3]{y^2 + 1}$, the ordinary Taylor series expansion of this function will do:

$$\sqrt[3]{t^2 + 1} = 1 + 0t + \frac{1}{3} t^2 + ...$$

satisfies $x - 1 \in m_p^2, x - 1 - \frac{1}{3} y^2 \in m_p^3, ...$

Aside on UFDness: In a Noetherian domain A, let $u \in A$ denote a unit. Then any nonzero $a \in A$ is a finite product:

$$a = uf_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$$

of a unit and irreducible elements $f_i \in A$, where $f \in A$ is irreducible if its only factors are u and uf. A UFD is defined by requiring that such products be unique, in the sense that the f_i are determined up to permutation and multiplication by units. There are several useful criteria for UFDness:

The Standard Criterion: A is a UFD if and only if each irreducible $f \in A$ generates a prime ideal (i.e. $f|gh \Rightarrow f|g$ or $f|h$). This is the familiar criterion used in elementary number theory to prove that the integers are a UFD.

Minimal Prime Criterion: A is a UFD if and only if every minimal prime ideal is a principal ideal.

Proof: A minimal prime ideal is a non-zero prime ideal that contains no other (non-zero) prime ideals. Suppose A is a UFD and $P \subset A$ is a minimal prime ideal. Let $0 \neq a \in P$. If we factor $a = uf_1^{m_1} \cdots f_n^{m_n}$, then (at least) one of the $f_i \in P$. But by the standard criterion $\langle f_i \rangle$ is a prime ideal, so $\langle f_i \rangle = P$ by minimiality. Conversely, if every minimal prime is principal, let $S \subset A$ be the subset generated by 1 and the irreducible elements $f \in A$ such that $\langle f \rangle$ is prime. As in the standard criterion, S is exactly the (multiplicative) set of elements of A with unique factorization. If $S = A - 0$, then A is a UFD. On the other hand if $S \neq A$ then A_S is not a field and then the non-zero (minimal) prime ideals in A_S correspond to (minimal) prime ideals $P \subset A$ with $P \cap S = \emptyset$. Such a $P = \langle f \rangle$ is principal, by assumption, and then f is irreducible, giving a contradiction since $f \notin S$.

Ratio Criterion: A is a UFD if and only if every "ratio" of principal ideals:

$$\langle f \rangle : \langle g \rangle = \{a \in A \mid ag \in \langle f \rangle\} \text{ is again a principal ideal}$$

Proof: If A is a UFD and $f = uf_1^{k_1}...f_n^{k_n}$ and $g = vf_1^{l_1}...f_n^{l_n}$ are their factorizations, $\langle f \rangle : \langle g \rangle = \langle e \rangle$ for $e = f_1^{\max(k_1 - l_1, 0)}...f_n^{\max(k_n - l_n, 0)}$. Conversely, suppose $f \in A$ is irreducible and $f|gh$. If the ratio property holds, then $\langle f \rangle : \langle g \rangle = \langle e \rangle$ for some $e \in A$. Notice that $f, h \in \langle f \rangle : \langle g \rangle$, so

$$f = ae, h = be \text{ and } eg = cf$$

for some elements $a, b, c \in A$. Since f is irreducible, either a is a unit, in which case $f|h$, or else e is a unit, and $f|g$ so the standard criterion applies.

Step 2: This is proved by induction using:

Weierstrass Preparation: If $F \in \mathbf{C}[[x_1, ..., x_n]]$ satisfies $F(0, ...0, x_n) \neq 0$, then there is a uniquely determined *unit* $U \in \mathbf{C}[[x_1, ..., x_n]]$ such that:

$$FU = x_n^d + R_1 x_n^{d-1} + ... + R_d \in \mathbf{C}[[x_1, ..., x_{n-1}]][x_n]$$

for some $d \geq 0$ and (uniquely determined) power series $R_i \in \mathbf{C}[[x_1, ..., x_{n-1}]]$.

(For a proof, see Zariski-Samuel *Commutative Algebra*)

Proof of Step 2: Given an irreducible $F \in \mathbf{C}[[x_1, ..., x_n]]$ dividing GH, then after (possibly reordering and) changing coordinates: $y_i = x_i + c_i x_n$, we can assume that $F(0, ..., 0, x_n) \neq 0$, and this can be done simultaneously for G and H, too. Now use Weierstrass Preparation: $FU, GV, HW \in \mathbf{C}[[y_1, ..., y_{n-1}]][x_n]$ for uniquely determined units U, V, W. The irreducibility of F implies FU is irreducible, and by Gauss' lemma and induction on n we know that $\mathbf{C}[[y_1, ..., y_{n-1}]][x_n]$ is a UFD, so

$$F|GH \Rightarrow FU|(GV)(HW) \text{ (in } \mathbf{C}[[x_1, ..., x_n]])$$

$$\Rightarrow FU|(GV)(HW) (\text{in } \mathbf{C}[[y_1, ..., y_{n-1}]][x_n])$$

$$\Rightarrow FU|GV \text{ or } FU|HW \text{ (in either ring)} \Rightarrow F|G \text{ or} F|H$$

Step 3: If $\widehat{A}$ is a UFD, then $A \subset \widehat{A}$ is a UFD.

Proof: We'll use the ratio criterion for UFDness. Given $f, g \in A$, then:

$$0 \to \langle f \rangle : \langle g \rangle \to A \xrightarrow{\cdot g} A/\langle f \rangle$$

is an exact sequence of A-modules, by definition of $\langle f \rangle : \langle g \rangle$. If we take completions, then the sequence remains exact, and becomes:

$$0 \to \widehat{\langle f \rangle : \langle g \rangle} \to \widehat{A} \xrightarrow{\cdot g} \widehat{A}/\langle f \rangle$$

Since $\widehat{A}$ is a UFD, $\widehat{\langle f \rangle : \langle g \rangle} = \langle f \rangle : \langle g \rangle = \langle \widehat{e} \rangle \subset \widehat{A}$ for some $\widehat{e} \in \widehat{A}$. But in a *local* ring (A, m) an ideal I is principal if and only if I/mI has rank 1 as a vector space over A/m (by Nakayama's lemma II!) And since:

$$\widehat{\langle f \rangle : \langle g \rangle}/\widehat{m} \cdot \widehat{\langle f \rangle : \langle g \rangle} = \langle f \rangle : \langle g \rangle / m \cdot \langle f \rangle : \langle g \rangle \text{ (is equal to its completion)}$$

it follows that $\langle f \rangle : \langle g \rangle \subset A$ is principal, as desired.

Exercises 10.

1. If $p \in X$ and $q \in Y$ are nonsingular points of quasi-projective varieties, prove that the point $(p, q) \in X \times Y$ is nonsingular.

2. (a) Prove Euler's relation. A polynomial $F \in \mathbf{C}[x_0, ..., x_n]$ is homogeneous of degree d if and only if:

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = dF$$

(b) If $F \in \mathbf{C}[x_0, ..., x_n]$ is an irreducible homogeneous polynomial, prove that $p \in V(F)$ is a non-singular point if and only if:

$$\nabla F(p) \neq (0, ..., 0)$$

Conclude that the Fermat hypersurfaces $V(x_0^d + ... + x_n^d)$ are all non-singular.

(c) If F is a *reducible* homogeneous polynomial, show that there is always a point $p \in V(F)$ such that:

$$\nabla F(p) = (0, ..., 0)$$

(d) Prove that, within the projective space $\mathbf{CP}^{\binom{n+d}{d}-1}$ of hypersurfaces $V(F) \subset \mathbf{CP}^n$ of degree d, there is a non-empty open subset $U \subset \mathbf{CP}^{\binom{n+d}{d}-1}$ such that $V(F) \in U$ if and only if F is irreducible and $V(F)$ is non-singular.

(e) If F is an irreducible quadric hypersurface (i.e. $d = 2$), describe the singular locus of $V(F)$. Find the complement $\mathbf{CP}^{\binom{n+2}{2}-1} - U$ for U in (d).

3. Prove that an algebraic group is non-singular. More generally, prove that a homogeneous space, that is, a variety X with a transitive action of an algebraic group G:

$$\sigma : G \times X \to X$$

is non-singular. Conclude that the Grassmannian $G(r, n)$ is non-singular.

4. For each of the degeneracy loci in the projective space of matrices:

$$D_r(m, n) = \{M \in \mathbf{CP}^{mn-1} \mid \mathrm{rk}(M) \leq r\}$$

prove that the singular locus satisfies:

$$\mathrm{Sing}(D_r(m, n)) = D_{r-1}(m, n)$$

and prove that at each singular point $p \in D_r(m, n)$, $\dim(m_p/m_p^2) = mn - 1$.

5. If $X \subset \mathbf{CP}^n$ is an embedded projective variety and $p \in X$, define:

$$\Theta_p(X) := \bigcap_{\text{homog } F \in I(X)} V(\sum_{i=0}^{n} \frac{\partial F}{\partial x_i}(p)x_i) \subseteq \mathbf{CP}^n$$

(a) Prove that $\Theta_p(X) \subset \mathbf{CP}^n$ is a projective subspace passing through p, of dimension equal to $\dim(m_p/m_p^2)$.

$\Theta_p(X)$ is called the *projective tangent space at p*. Projective varieties $X, Y \subset \mathbf{CP}^n$ intersect *transversely* at $p \in X \cap Y$ if p is a non-singular point of both X and Y, and

$$\dim(\Theta_p(X) \cap \Theta_p(Y)) = \dim(X) + \dim(Y) - n$$

(i.e. if the projective tangent spaces intersect transversely)

(b) If X and Y intersect transversely at p, show that the component:

$$p \in Z \subset X \cap Y$$

of $X \cap Y$ containing p is non-singular at p.

6. Bertini's Theorem. If $X \subset \mathbf{CP}^n$ is a non-singular embedded projective variety of dimension ≥ 2, consider the subset:

$$U \subseteq (\mathbf{CP}^n)^* \cong \mathbf{CP}^n \text{ (the projective space of hyperplanes in } \mathbf{CP}^n\text{)}$$

defined by the property:

$$H \in U \Leftrightarrow H \cap X \text{ is irreducible and non-singular}$$

Prove that U is a non-empty Zariski open subset of $(\mathbf{CP}^n)^*$.

Hints: Consider the (projective variety!) subset:

$$TX := \{(p, H) \mid p \in X, \Theta_p(X) \subseteq H\} \subset \mathbf{CP}^n \times (\mathbf{CP}^n)^*$$

The projection onto $\mathbf{CP}^n$ is X with fibers isomorphic to $\mathbf{CP}^{n-\dim(X)-1}$ so TX is irreducible, of dimension $n-1$. Therefore its projection to $(\mathbf{CP}^n)^*$ has (closed) image of dimension $\leq n-1$, and any U contained in the complement of the image will give non-singular $H \cap X$. But what about irreducibility?

Prove the theorem with "hyperplane" replaced by "hypersurface of deg d."

7. If $X \subset \mathbf{CP}^n$ is a variety (maybe singular) of dimension m, show that there is a non-empty open subset:

$$U \subset (\mathbf{CP}^{n*})^m$$

such that, if $(H_1, ..., H_m) \in U$, then:

$$(H_1 \cap ... \cap H_m) \cap \mathrm{Sing}(X) = \emptyset$$

and

$$(H_1 \cap ... \cap H_m) \cap X$$

is transverse at the non-singular locus of X, and consists of d points, where:

$$H_X(t) = \frac{d}{m!} t^m + ...$$

is the Hilbert polynomial of $X \subset \mathbf{CP}^n$.